5 Estimation

We observe i.i.d. RVs $X_1, X_2, \dots$ and want to estimate

$$\mu \quad \text{and} \quad \sigma^2.$$

Unbiased estimators are

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

How good are estimates obtained with them?

5.1. Interval Estimates

As estimates we provide intervals (a,b) such that

$$\mu \in (a,b) \quad \text{or} \quad \sigma^2 \in (a,b)$$

with a "high degree of confidence".

Assumption: $X_1, X_2, \ldots$ are $N(\mu, \sigma^2)$ i.i.d.

(we can sometimes drop this assumption due to the CLT)

Not with "high probability" because we don't know the distribution of μ or σ^2.

Cases:

- σ^2 is known, μ is unknown
- both σ^2 and μ are unknown, interested in μ
- σ^2 is unknown (μ is w/o interest)

Interval Types:

- two-sided intervals: (a, b)
- lower intervals: $(-\infty, b)$
- upper intervals (a, ∞)

5.2 Estimation of two-sided Intervals

Suppose we want to be "95% confident" that $\mu \in (a,b)$.

How should we choose a, b?

Use Case 1: How good is, for given values $x_1, \dots, x_n$ for $X_1, \dots, X_n$, the average $\bar{X}$ as an estimate of the mean $E[X]$?

Idea: Design functions

$$a(x), \quad b(x)$$

with corresponding RVs $A = a(\bar{X})$, $B = b(\bar{X})$ s.th.

$$P[A \leq \mu \leq B] = 95\%$$

Terminology: $5\% = 100\% - 95\%$ is the confidence level.

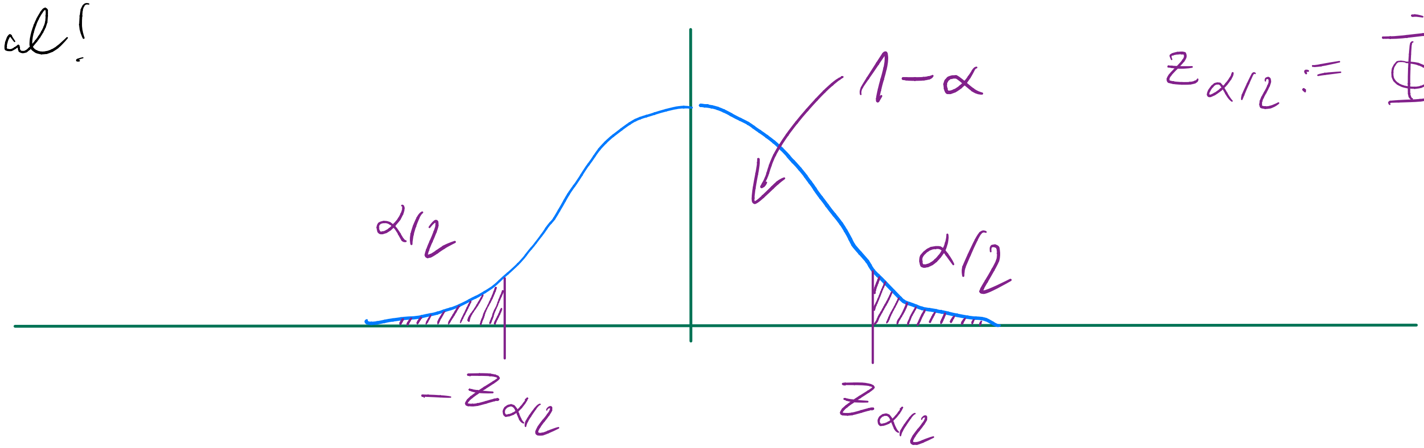
How Choose a, b?

We consider the problem to determine constants a, b s.th.

$$P[a \le Z \le b] = 1 - \alpha, \qquad Z \sim \mathcal{N}(0,1)$$

Trivial!

$$z_{\alpha/2} := \Phi^{-1}\left(1 - \alpha/2\right)$$



Choose $a = -z_{\alpha/2}$, $b = z_{\alpha/2}$

In general define:

$$z_\alpha := \Phi^{-1}(1-\alpha).$$

Next suppose, $X \sim N(\mu, 1)$, but μ unknown

$$\Rightarrow X - \mu \in N(0,1) \Rightarrow \mu - X \in N(0,1)$$

$$1-\alpha = P\left[-z_{\alpha/2} \leq \mu - X \leq z_{\alpha/2}\right]$$

$$= P\left[\underbrace{X - z_{\alpha/2}}_{a(X)} \leq \mu \leq \underbrace{X + z_{\alpha/2}}_{b(X)}\right]$$

Note: The interval boundaries are RV, not the mean μ.

Finally, $X_i \sim N(\mu, \sigma^2) \Rightarrow \bar{X} \sim N(\mu, \frac{1}{n}\sigma^2)$

$$\Rightarrow \bar{X} - \mu \sim N(0, \tfrac{1}{n}\sigma^2) \Rightarrow \mu - \bar{X} \sim N(0, \tfrac{1}{n}\sigma^2)$$

$$\Rightarrow \sqrt{n}\,\frac{\mu - \bar{X}}{\sigma} \sim N(0,1)$$

$$\Rightarrow 1 - \alpha = P\left[-z_{\alpha/2} \leq \sqrt{n}\,\frac{\mu - \bar{X}}{\sigma} \leq z_{\alpha/2}\right]$$

$$\Rightarrow \quad 1-\alpha = P\left[-z_{\alpha/2} \leq \sqrt{n}\,\frac{\mu-\bar{X}}{\sigma} \leq z_{\alpha/2} \right]$$

$$= P\left[-\frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \mu-\bar{X} \leq \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right]$$

$$= P\left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \mu \leq \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right]$$

Proposition: Let $\bar{X} \sim N(\mu, \frac{1}{n}\sigma^2)$. Let

- $a(x) = x - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}$
- $b(x) = x + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}$.

Then

$$P\left[a(\bar{X}) \leq \mu \leq b(\bar{X}) \right] = 1-\alpha$$

We say that $(a(\bar{x}), b(\bar{x}))$ is a $(1-\alpha)\cdot 100\%$ confidence interval for the mean, for a given $\bar{x}$.