Example 61: We want to send signals 0, 1 over a channel with noise. We encode

$$0 \text{ as } -2$$
$$1 \text{ as } 2.$$

The receiver sees $R = x + N$, $N \sim \mathcal{N}(0,1)$ and decodes

$$R \geq 0.5 \text{ as } 1$$
$$R < 0.5 \text{ as } 0$$

What is the probability of an error in each case?

Sender: 0 as -2

1 as 2

$R = x + N$

Receiver: $R \geq 0.5$ as 1

$R < 0.5$ as 0

Sender: 0 as -2 $\quad R = x + N$ $\quad$ Receiver: $R \geq 0.5$ as 1

1 as 2 $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $R < 0.5$ as 0

Error in receiving 1:

$$P[R < 0.5 \mid S = 1] = P[x + N < 0.5 \mid x = 2]$$

$$= P[N < -1.5] = P[N > 1.5] = 1 - P[N \leq 1.5]$$

$$= 1 - \Phi(1.5)$$

Error in receiving 2:

$$P[R \geq 0.5 \mid S = 0] = P[x + N \geq 0.5 \mid x = -2]$$

$$= P[N \geq 2.5] = 1 - P[N \leq 2.5]$$

$$= 1 - \Phi(2.5)$$

Example 62: Suppose the height of European males is normally distributed with mean $\mu = 177.6$ cm and standard deviation $\sigma = 4$ cm.

- What is the probability that among two brothers the older is at least 2 cm taller than the younger (assuming independence of their height)?

Example 62: Suppose the height of European males is normally distributed with mean $\mu = 177.6$ cm and standard deviation $\sigma = 4$ cm.

- What is the probability that among two brothers the older is at least 2 cm taller than the younger (assuming independence of their height)?

Let H be the height of European men and H_1, H_2 two independent copies. Let $D := H_1 - H_2$. We are interested in

$$P[D \geq 2].$$

We know that

$$H_1, H_2 \sim N(\mu, \sigma^2) \Rightarrow -H_2 \sim N(-\mu, \sigma^2)$$

$$\Rightarrow D = H_1 - H \sim N(\mu - \mu, \sigma^2 + \sigma^2)$$

$$= N(0, 2\sigma^2)$$

Then

$$P[D \geq 2] = P\left[\frac{1}{\sqrt{2}\,8} D \geq \frac{2}{\sqrt{2}\,8}\right] = P\left[Z \geq \frac{2}{\sqrt{2}\,8}\right]$$

$$= 1 - P\left[Z \leq \frac{2}{\sqrt{2}\,8}\right] = 1 - \Phi\left(\frac{2}{\sqrt{2}\cdot 4}\right)$$

$$\approx 1 - \Phi(0.3536) = 0.3632$$

The 68-95-99.7 Rule

Let $Z \sim \mathcal{N}(0,1)$. Then

$$P[-1 \le Z \le 1] \approx .68$$

$$P[-2 \le Z \le 2] \approx .95$$

$$P[-3 \le Z \le 3] \approx .997$$

For $X \sim \mathcal{N}(\mu, \sigma^2)$, this means

$$P[\mu - \sigma \le X \le \mu + \sigma] \approx .68$$

$$P[\mu - 2\sigma \le X \le \mu + 2\sigma] \approx .95$$

$$P[\mu - 3\sigma \le X \le \mu + 3\sigma] \approx .997$$

That is

68%	of all values are within	1	standard deviation(s) of the mean
95%		2	
99.7%		3	