3.5 The Normal Distribution

Preliminaries:

What is $$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx \quad ?$$

Difficult! There is no primitive for the integrand.

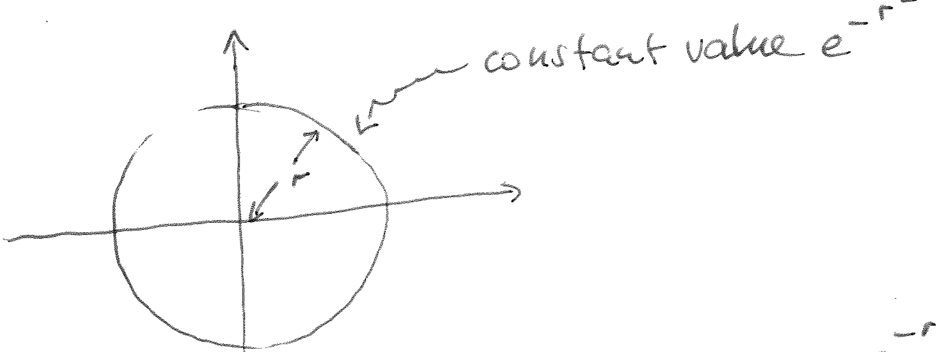
Better:

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy$$

The integrand is constant on every circle around $(0,0)$. On a circle with radius r it has the value e^{-r^2}. Each such circle has length $2\pi r$.



For each $r > 0$, we have to integrate $2\pi r e^{-r^2}$.

$$I^2 = \int_0^\infty 2\pi r \cdot e^{-r^2}\, dr = \pi \left[-e^{-r^2} \right]_0^\infty = \pi$$

Hence, $\quad I = \sqrt{\pi}$

What is $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$?

We write this as

$$\int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}}\right)^2} dx \qquad\qquad y = \frac{x}{\sqrt{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow dx = \sqrt{2}\, dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \sqrt{2\pi}$$

Consequence:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

is the pdf of a continuous distribution,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$

is the corresponding cdf.

Suppose $Z \sim \Phi$. What are mean and variance?

$$\mu = \int_{-\infty}^{\infty} x \, \varphi(x) \, dx$$

We notice:

- φ is symmetric wrt 0, that is,
  $\varphi(x) = \varphi(-x)$ (φ is an <u>even</u> function)
- $x\,\varphi(x)$ is <u>odd</u>, that is
  $x\,\varphi(x) = -(-x)\,\varphi(-x)$

Hence, $\mu = 0$

Alternatively, we calculate

$$\mu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[- e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} = 0$$

$$E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot x e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[-x \cdot e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(0 + \sqrt{2\pi} \right) = 1$$

Hence,

$$Var(Z) = E[Z^2] - E[Z]^2 = 1 - 0 = 1$$

We say that φ and Φ are the pdf and the cdf of the normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$. This is also called the standard normal distribution, written with the symbols

$$\mathcal{N}(0,1)$$

Let $Z \sim \mathcal{N}(0,1)$ and let $X := \sigma Z + \mu$.

Then

$$E[X] = E[\sigma Z + \mu]$$

$$= \sigma E[Z] + \mu = \sigma \cdot 0 + \mu = \mu$$

$$Var(X) = Var(\sigma Z + \mu)$$

$$= \sigma^2 Var(Z) = \sigma^2 \cdot 1 = \sigma^2$$

What are the cdf and pdf of X?

$$F_X(x) = P[X \leq x] = P[\sigma Z + \mu \leq x]$$

$$= P\left[Z \leq \frac{x-\mu}{\sigma}\right] = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Therefore,

$$f_X(x) = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This is the density of the normal distribution with mean μ and variance σ^2, denoted as

$$\mathcal{N}(\mu, \sigma^2)$$

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$.

Let $a \in \mathbb{R}$, $b > 0$. What is the distribution of

$$Y = aX + b \text{ ?}$$

We know $X \sim \sigma Z + \mu \quad (Z \sim \mathcal{N}(0,1))$

Hence,
$$\begin{aligned} Y &\sim a(\sigma Z + \mu) + b \\ &= a\sigma Z + a\mu + b \\ &\sim \mathcal{N}(a\mu + b, a^2\sigma^2) \end{aligned}$$

That is, the linear transformation of a normal RV is normal.

<u>Reduction to the standard normal</u>

If $X \sim \mathcal{N}(\mu, \sigma)$, then

$$\frac{1}{\sigma} X - \frac{\mu}{\sigma} = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

Hence

$$P[X \leq x] = P\left[\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right] = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Therefore, we can compute probabilities for X using the cdf Φ of the standard normal $\mathcal{N}(0,1)$

Φ cdf of $\mathcal{N}(0,1)$: has tables and is implemented in software

Tables are often only for $\Phi(x)$, $x > 0$.
However, for $x > 0$ we have

$$\Phi(-x) = P[Z \leq -x] = 1 - P[Z \geq x]$$
$$= 1 - \Phi(x)$$

because the pdf φ is even (= symmetric around 0).

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$P[X \leq x] = P[\sigma Z + \mu \leq x] = P[Z \leq \frac{x-\mu}{\sigma}]$$
$$= \Phi[\frac{x-\mu}{\sigma}]$$

<u>Example 6.1</u>: We want to send signals 0, 1 over a channel with noise. We encode

0 as -2

1 as 2.

The receiver sees $R = X + N$, $N \sim \mathcal{N}(0,1)$, and decodes

$R \geq 0.5$ as 1

$R < 0.5$ as 0

What is the probability of an error in each case?

Error in receiving 1:

$$P[R < 0.5 \mid S = 1] = P[x + N < 0.5 \mid x = 2]$$

$$= P[N < -1.5] = P[N > 1.5]$$

$$= 1 - P[N \leq 1.5] = 1 - \Phi(1.5)$$

$$= 1 - 0.9332 = 0.0668$$

Error in receiving 0:

$$P[R \geq 0.5 \mid S = 0] = P[x + N \geq 0.5 \mid x = -2]$$

$$= P[N \geq 2.5] = 1 - P[N \leq 2.5]$$

$$= 1 - \Phi(2.5) = 1 - 0.9938 = 0.0062$$

Sums of normal RVs

Proposition: Suppose X_1, X_2 are independent,

$X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

Then

$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

That is, the sum of two independent normal RVs is a normal RV.

Expected value and variance of the sum are determined according to the rules for $E[\cdot]$ and $Var(\cdot)$

Example 62: Suppose the height of European males is a normal with $\mu = 177.6$ cm and $\sigma = 4$ cm. What is the probability that among two brothers the older is at least 2cm taller than the younger (assuming independence)

Let X_1, X_2 be the height of the first and second brother. Then

$$P[X_1 \geq X_2 + 2] = P[X_1 - X_2 \geq 2]$$

$$= P[Y \geq 2] = 1 - P[Y \leq 2] \quad \text{where } Y = X_1 - X_2$$

We have that $Y \sim N(0, 2\sigma^2) = N(0, 32)$

$$P[Y \leq 2] = P[\sqrt{32}\, Z \leq 2] = P[Z \leq \tfrac{2}{\sqrt{32}}]$$

$$= \Phi\left(\tfrac{2}{\sqrt{32}}\right)$$

$$P[Y \geq 2] = 1 - \Phi\left(\tfrac{2}{\sqrt{32}}\right) \approx 1 - \Phi(0.3536)$$

$$\approx 1 - 0.6368 = 0.3632$$

The 68 – 95 – 99.7 Rule

Let $Z \sim N(0,1)$. Then

$$P[-1 \leq Z \leq 1] \approx .68$$

$$P[-2 \leq Z \leq 2] \approx .95$$

$$P[-3 \leq Z \leq 3] \approx .997$$

That is

68 % of all values are with 1 standard deviation of the mean

95 % —— " —— 2 —— " ——

99.7 % —— " —— 3 —— " ——

For $X \sim N(\mu, \sigma^2)$, this means

$$P[\mu - \sigma \leq X \leq \mu + \sigma] \leq .68$$

$$P[\mu - 2\sigma \leq X \leq \mu + 2\sigma] \leq .95$$

$$P[\mu - 3\sigma \leq X \leq \mu + 3\sigma] \leq .997$$