We have seen that X and Y have the same density, which we denote as f. Since $d(x,y) = g(\sqrt{x^2+y^2})$, we have

$$g(\sqrt{x^2+y^2}) = f(x) \cdot f(y) \qquad \text{f.a. } x,y \in \mathbb{R}$$

For nonnegative x, y, we have $x = \sqrt{x^2}$ and $y = \sqrt{y^2}$.
We can then rewrite this equation as

$$g(\sqrt{x^2+y^2}) = f(\sqrt{x^2}) \cdot f(\sqrt{y^2}).$$

We then see that the function $g(\sqrt{\cdot})$ turns sums of squares into products of values of $f(\sqrt{\cdot})$.

We also have for nonnegative x that

$$g(x) = g(\sqrt{x^2+0}) = f(x) \cdot f(0) = K \cdot f(x)$$

with $K = f(0)$, or $f(x) = \frac{1}{K} g(x)$.

From the equation

$$g(\sqrt{x^2+y^2}) = f(\sqrt{x^2}) \cdot f(\sqrt{y^2})$$

we then conclude that

$$g(\sqrt{x^2+y^2}) = f(\sqrt{x^2}) \cdot f(\sqrt{y^2}) = \frac{1}{K^2} g(\sqrt{x^2}) \cdot g(\sqrt{y^2})$$

for all $x, y \in \mathbb{R}$. Since every nonnegative number is the square of some number, this also shows that for all $u, v \in \mathbb{R}_0^+$ we have that

$$g(\sqrt{u+v}) = \frac{1}{K^2} g(\sqrt{u}) \cdot g(\sqrt{v})$$

Multiplying this by $\frac{1}{K^2}$ yields

$$\frac{1}{K^2} g(\sqrt{u+v}) = \frac{1}{K^2} g(\sqrt{u}) \cdot \frac{1}{K^2} g(\sqrt{v}).$$

With $h(u) := \frac{1}{K^2} g(\sqrt{u})$, this is

$$h(u+v) = h(u) \cdot h(v), \qquad u, v \in \mathbb{R}_0^+$$

From

$$h(u+v) = h(u) \cdot h(v), \qquad u, v \in \mathbb{R}_0^+$$

we conclude, based on our study of exponential functions) that

$$h(u) = a^u \text{ for some } a > 0.$$

Since $h(u) = \frac{1}{K^2} g(\sqrt{u})$, we have

$$\frac{1}{K^2} g(\sqrt{u}) = a^u \quad , \qquad u \geq 0$$

We also had $g(x) = K \cdot f(x)$. Thus

$$a^x = \frac{1}{K^2} g(\sqrt{x}) = \frac{1}{K^2} \cdot K \cdot f(\sqrt{x}) = \frac{1}{K} f(\sqrt{x})$$

$$\Rightarrow f(\sqrt{x}) = K \cdot a^x$$

$$\Rightarrow f(x) = f(\sqrt{x^2}) = K \cdot a^{x^2} \quad , \quad x \geq 0$$

So, we have

$$f(x) = K \cdot a^{x^2} \quad , \text{ f.a. } x \geq 0.$$

What about negative x? Note that

$$f(x) \cdot f(0) = g(\sqrt{x^2 + 0^2}) = g(\sqrt{(-x)^2 + 0^2})$$

$$= f(-x) \cdot f(0),$$

hence

$$f(x) = f(-x), \qquad \text{f.a. } x \in \mathbb{R},$$

Therefore,

$$f(x) = K \cdot a^{x^2} \qquad \text{f.a. } x \in \mathbb{R},$$

So, our marginal density $f = f_X = f_Y$ has the form

$$f(x) = K a^{x^2}.$$

What does this mean for a and K? This can be concluded from the requirements of a density:

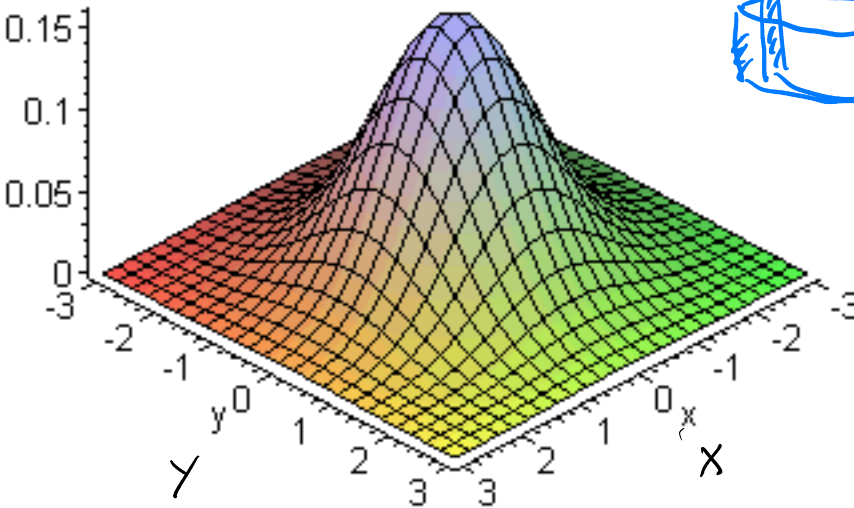
$$f \geq 0 \quad \text{and} \quad \int_{\mathbb{R}} f(x)\, dx = 1.$$

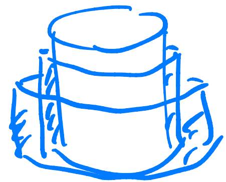
The first condition is obviously met ($a^{x^2} > 0$, f.a. $x \in \mathbb{R}$).
The second implies that $a < 1$, since otherwise $\lim_{x \to \infty} a^{x^2} = \infty$.
Therefore let $\alpha := \log \frac{1}{a}$, which is greater 0.
Then

$$f(x) = K e^{-\alpha x^2}.$$

Bivariate Normal

$$d(x,y) = f(x) \cdot f(y)$$





Now, K and α are tied together by the constraint that

$$K \int_{\mathbb{R}} e^{-\alpha x^2} \, dx = 1.$$

Determining this constraint is made difficult by the fact that antiderivatives of e^{x^2} cannot be represented by an elementary expression.

However, our original interest was not in the density f, but in $d(x,y) = f(x) \cdot f(y)$. What can we deduce from

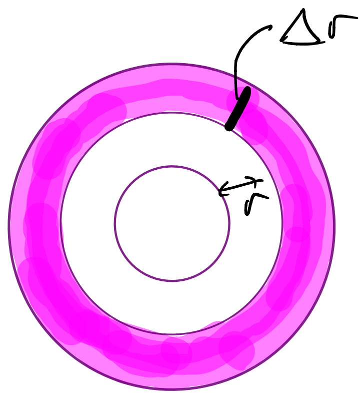
$$1 = \int_{\mathbb{R}} \int_{\mathbb{R}} d(x,y) \, dx \, dy = K^2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\alpha x^2} e^{-\alpha y^2} \, dx \, dy$$

$$= K^2 I_2 \quad ?$$

First, we concentrate on I_2:

$$I_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\alpha x^2} e^{-\alpha y^2}\,dx\,dy = \iint_{\mathbb{R}\,\mathbb{R}} e^{-\alpha(x^2+y^2)}\,dx\,dy$$

The integrand depends only on the distance r of its argument from the origin: if (x,y) is on a circle with radius r, then the integrand has value $e^{-\alpha r^2}$.



A circle with width Δr and radius r has approximately area $2\pi r \cdot \Delta r$ and contributes approximately a value

$$e^{-\alpha r^2} \cdot 2\pi r\,\Delta r$$

to the integral. With $\Delta r \to 0$ this gives

$$I_2 = \int_0^\infty 2\pi r\, e^{-\alpha r^2}\,dr.$$

This can be evaluated.

The next two pages are an alternative derivation of the equality

$$\iint_{\mathbb{R}\,\mathbb{R}} d(x,y)\,dx\,dy = K^2 \int_0^\infty 2\pi r\, e^{-\alpha r^2}\,dr$$

which takes account of questions during the lecture.

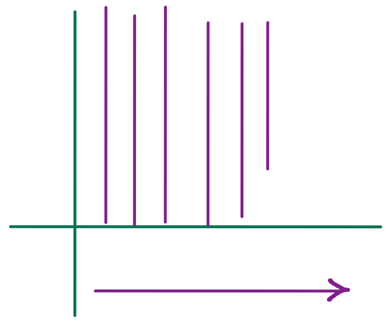
More information can be found, for instance on Wikipedia, in articles on

- shell integration
- polar coordinates
- Gauss integral

Note that this is not an exam subject but only intended to help you understand the background of the normal distribution.

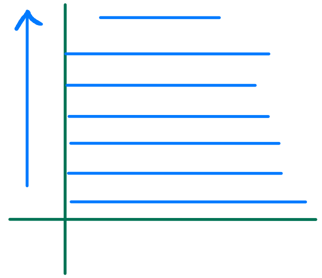
Integrating a Function with Rotational Symmetry

How can we integrate in an easy manner a function that depends only on the distance from the origin?



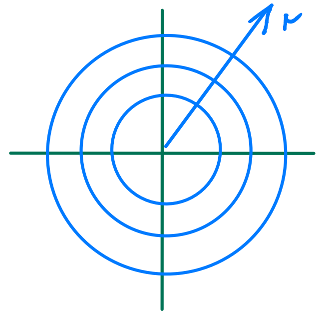
In the past we have integrated a function $f(x,y)$ either

- by first integrating over y for fixed x, then the results over x, or



- by first integrating over x for fixed y, then the results over y

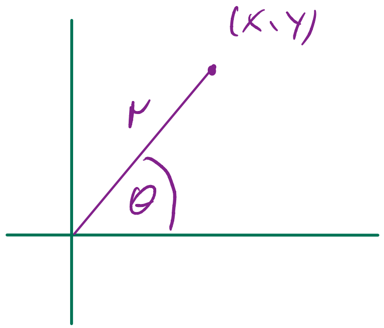
Alternatively we can integrate, for fixed distance $r \geq 0$, over all angles θ, $0 \leq \theta \leq 2\pi$, and then integrate the results over r.



The result of integrating over θ has to be multiplied by $2\pi r$, to take into account the length of the circle over which we integrated.

So,
$$\int_{\mathbb{R}}\int_{\mathbb{R}} d(x,y)\, dx\, dy$$

The density at point (x,y) with distance r and angle θ is $Ke^{-\alpha r^2}$.



$$= \int_0^\infty \int_0^{2\pi} d(r,\theta)\; r\, d\theta\, dr$$

$$= \int_0^\infty \int_0^{2\pi} K^2 e^{-\alpha r^2}\; r\, d\theta\, dr$$

$$= \int_0^\infty K^2 e^{-\alpha r^2} \cdot r \int_0^{2\pi} 1\; d\theta\, dr$$

$$= K^2 \int_0^\infty e^{-\alpha r^2} \cdot 2\pi r\; dr$$

The density is constant on every circle.
Over the circle of radius r, it contributes
$$2\pi r \cdot e^{-\alpha r^2},$$

i.e., function value times length of circle line.

I_2 can be evaluated using the substitution rule:

Here, f, g are just symbols, not the functions we had before!

$$\int_0^\infty 2\pi r e^{-\alpha r^2} dr = C \int_0^\infty f(g(r)) \cdot g'(r)\, dr$$

$$= -\frac{\pi}{\alpha} \int_0^\infty \underbrace{\left(-e^{-\alpha r^2}\right)}_{f,\ g} \underbrace{2\alpha r}_{g'}\, dr$$

$$= -\frac{\pi}{\alpha} \int_{g(0)}^{g(\infty)} f(z)\, dz$$

$$= -\frac{\pi}{\alpha} \int_{g(0)}^{g(\infty)} -e^{-z}\, dz$$

$$= -\frac{\pi}{\alpha} \left[e^{-z}\right]_{g(0)}^{g(\infty)} = -\frac{\pi}{\alpha} \left[e^{-z}\right]_0^\infty$$

$$= -\frac{\pi}{\alpha}(0-1) = \frac{\pi}{\alpha}$$

We had the constraint $K^2 I_2 = 1$.

Hence, $K^2 \frac{\pi}{\alpha} = 1$ and therefore $K = \sqrt{\frac{\alpha}{\pi}}$. Thus

$$\boxed{f(x) = \frac{\sqrt{\alpha}}{\sqrt{\pi}} e^{-\alpha x^2}}$$

is the pdf of X and Y.

Mean and Variance of f:

$$f(x) = \frac{\sqrt{\alpha}}{\sqrt{\pi}} e^{-\alpha x^2}$$

$\boxed{\text{Mean:}}$ Clearly, f is symmetric around 0, that is, $f(x) = f(-x)$.

Hence, the mean μ, which is the center of gravity, is 0.

$\boxed{\text{Variance:}}$ We apply integration by parts

$$\int f g' = f g - \int f' g$$

$$\sigma^2 = \int_{\mathbb{R}} (x-\mu)^2 f(x)\, dx = \int_{\mathbb{R}} x^2 f(x)\, dx$$

$$= K \int_{\mathbb{R}} x^2 e^{-\alpha x^2} dx = K \int_{\mathbb{R}} \underbrace{\left(-\frac{1}{2\alpha} x\right)}_{f} \underbrace{\left(-2\alpha x \cdot e^{-\alpha x^2}\right)}_{g'} dx$$

$$= K \left(\left[\underbrace{\left(-\frac{1}{2\alpha} x\right)}_{f} \underbrace{\left(e^{-\alpha x^2}\right)}_{g} \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} \underbrace{-\frac{1}{2\alpha}}_{f'} \underbrace{e^{-\alpha x^2}}_{g} dx \right)$$

$$= \frac{1}{2\alpha} K \int_{\mathbb{R}} e^{-\alpha x^2} dx = \frac{1}{2\alpha}$$

General Form of Normal Density (with $\mu = 0$)

So, $\sigma^2 = \frac{1}{2\alpha} \implies \alpha = \frac{1}{2\sigma^2}$

$$\implies K = \frac{\sqrt{\alpha}}{\pi} = \frac{1}{\sqrt{2\sigma^2}} \cdot \frac{1}{\pi} = \frac{1}{\sqrt{2\pi} \cdot \sigma}$$

Hence,

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

This is a density with mean $\mu = 0$ and variance σ^2.

General Form of Normal Density With Arbitrary Mean

Imagine the star we are observing is not at position $(0,0)$, but (μ,ν). Then the error densitity would depend on the distance from that point, that is, on

$$\sqrt{(x-\mu)^2+(y-\nu)^2}.$$

In that case the marginals would have the form

$$\frac{1}{\sqrt{2\pi}\,\sigma}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

or the analogue one with ν. We say that a RV with that density has a normal distribution $N(\mu,\sigma^2)$. In the case of $N(0,1)$, we speak of the standard normal, which has density

$$\phi(x)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}.$$

Cumulative Distribution of the Standard Normal

The cumulative distribution (cdf) of the standard normal is denoted as Φ and satisfies

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx.$$

However, Φ cannot be represented in elementary terms (i.e., there is no formula). It can be computed approximately by numeric integration. Implementations exist in statistical libraries (R, Java packages). There are also tables.

Often, given probability p, one is interested in the x such that

$$\Phi(x) = P[X \leq x] = p.$$

that is

$$x = \Phi^{-1}(p).$$

Tables of the Normal

Tables are the traditional means to look up values of Φ.

To avoid redundancy, they only contain values $\Phi(x)$ for $x \geq 0.5$.

The symmetry of ϕ is reflected by Φ as

$$\Phi(-x) = 1 - \Phi(x), \qquad x \geq 0,$$

since for an $N(0,1)$-distributed RV Z we have

$$\Phi(-x) = P[Z \leq -x] \overset{\text{symmetry of } \phi}{=} P[Z > x] = 1 - P[Z \leq x] = 1 - \Phi(x)$$

Properties of Normal Distributions

We say that X is normally distributed if

$X \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$.

Proposition: Let X, Y be normally distributed and independent, $a > 0$, $b \in \mathbb{R}$. Then

- $aX + b$
- $X + Y$

are normally distributed

Proof (Idea): If $X \sim f$ (density f), then $aX + b \sim g$ where $g(y) = f\left(\frac{y-b}{a}\right)$, because $y = ax + b \Rightarrow x = \frac{y-b}{a}$

Check: if f is a normal density, then so is g.

The second part is more difficult, needs convolution

Corollary: $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$, $a, b \in \mathbb{R}$. Then

- $aX + b \sim N(a\mu_X + b, a^2 \sigma_X^2)$
- $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

We denote RVs that are $N(0,1)$-distributed as Z.

Proposition: Let $Z \sim N(0,1)$, $X \sim N(\mu, \sigma^2)$. Then

- $\sigma Z + \mu \sim N(\mu, \sigma^2)$
- $\dfrac{X - \mu}{\sigma} \sim N(0,1)$