Example 58 The number of customers in a bar is on average 4 per hour. What is the probability that there are no more than 3 in 2 hours?

$\mathcal{C}$: # customers / hour $\sim$ Pois(4)

$\mathcal{C}_2$: # customers / 2 hours $\sim$ Pois(4 + 4)
= Pois(8)

$P[\mathcal{C}_2 \leq 3]$

$\mathcal{C}_{\text{hour } 1} + \mathcal{C}_{\text{hour } 2}$

indep.

Example 58 The number of customers in a bar is on average 4 per hour. What is the probability that there are no more than 3 in 2 hours?

Let X_1 = # customers in 1^{st} hour

X_2 = # customers in 2^{nd} hour

Remember the Poisson story

- X_1, X_2 independent $\Rightarrow$ $X_1 + X_2$ Poisson
- Rate of $X_1 + X_2$ is $4+4 = 8$ in 2 hours

Reproductive property of the Poisson

$$P[X_1 + X_2 \leq 3] = \sum_{k=0}^{3} e^{-8} \frac{8^k}{k!} = 0.423$$

The Poisson distribution is reproductive in the following sense.

Proposition: Let $X_1 \sim Pois(\lambda_1)$, $X_2 \sim Pois(\lambda_2)$, X_1, X_2 ind.

Then

$$X_1 + X_2 \sim Pois(\lambda_1 + \lambda_2)$$

Proof (by story): X_1 counts events of type 1 with rate λ_1, X_2 events of type 2 with rate λ_2, which happen independently. What is the rate at which both kinds of events happen? Clearly, $\lambda_1 + \lambda_2$.

Alternative proofs by calculation (see lecture notes of 19/20 or scriptum.

Suppose there is a shop visited by λ costumers per hour.
Suppose that a fraction of p are female and of $(1-p)$ are male.
How is the number of female customers distributed?

F # female customers per hour

$F \sim \text{Pois}(p\lambda)$

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Suppose that a fraction of p are female and of $(1-p)$ are male.
How is the number of female customers distributed?

Answer: $\text{Pois}(p\lambda)$, since $p\lambda$ is their rate of arrival.

3.5 The Normal Distribution

Since the 17^{th} century astronomers developed more and more precise instruments to measure the position of stars. At the same time they noticed that their measurements always contained errors and they were keen to understand how those errors were distributed.

In 1809 Carl Friedrich Gauss published his method of least squared errors and related it in passing to a distribution that since then is known as the Gaussian.

The British astronomer John Herrschel in 1850 demonstrated how this distribution arises from simple assumptions about the underlying principles. We give here a derivation from the same assumptions with elementary arguments.

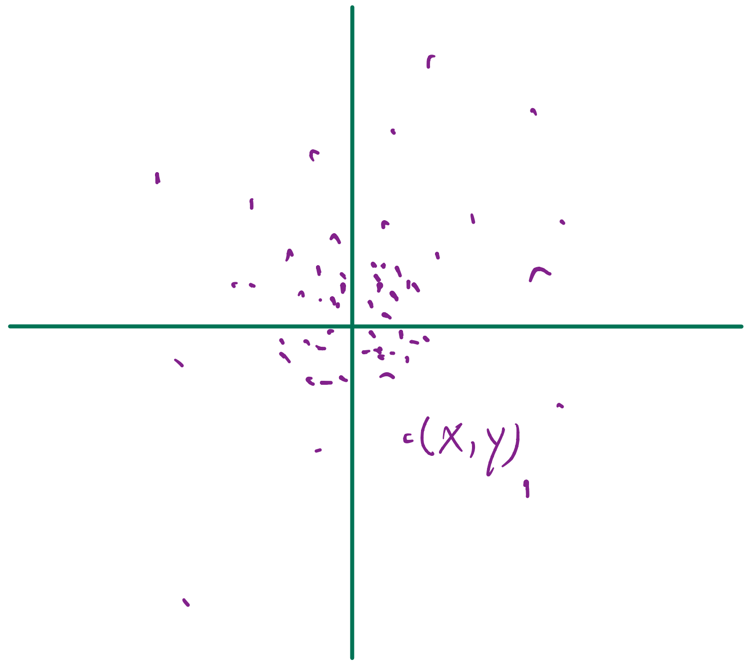
Astronomers determined the coordinates of an object in the sky with telescopes that can be positioned in horizontal and vertical direction. The object would have a unique position, but the astronomer would measure a (slightly) different one.

Let us assume that positions are described as (x,y)-coordinates and that the exact position of the object of our interest is the origin $(0,0)$. The (x,y)-measurements by an astronomer can be seen as the values of random variables X, Y, which have a joint distribution.

Let $d(x,y)$ be the density of that joint distribution. This is then a probability distribution of errors, since every measurement other than $(0,0)$ is erroneous.

What are reasonable assumptions about d?

Herrschel proposed two:



- The probability of errors (x, y) should not depend on their direction from the origin, but only on the distance from the origin.

- Errors along the x-axis should be independent of errors along the y-axis. (Astronomers have two distinct mechanisms for the calibration of their telescopes in each direction.)

What does this mean mathematically?

- The distance of (x,y) to the origin is $\sqrt{x^2+y^2}$ (Pythagoras!). Therefore, there is a function $g: \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ such that

$$d(x,y) = g(\sqrt{x^2+y^2})$$

- Let f_x, f_y be the marginal densities of d. Then the independence of x and y implies

$$d(x,y) = f_x(x) \cdot f_y(y).$$

Let us first investigate the relationship between f_X and f_Y.

Since d depends on the distance of the argument from the origin, we have

$$d(x,0) = g\left(\sqrt{x^2+0}\right) = g\left(\sqrt{0+x^2}\right) = d(0,x)$$

Hence,

$$f_X(x) \cdot f_Y(0) = d(x,0) = d(0,x) = f_X(0) \cdot f_Y(x)$$

and therefore

$$f_Y(x) = \frac{f_Y(0)}{f_X(0)} \cdot f_X(x).$$

Both f_X and f_Y are densities. Thus,

$$1 = \int_{\mathbb{R}} f_Y(x)\,dx = \frac{f_Y(0)}{f_X(0)} \cdot \int_{\mathbb{R}} f_X(x)\,dx = \frac{f_Y(0)}{f_X(0)} \cdot 1 = \frac{f_Y(0)}{f_X(0)}$$

We conclude that $f_Y(x) = f_X(x)$ f.a. $x \in \mathbb{R}$