3.4 Exponential Random Variables

A RV X is exponentially distributed with parameter $\lambda > 0$ if it has the density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Models the time between independent events that happen at a constant average rate λ.

Examples: Time between
- a car passing after the previous
- radioactive particles decaying
- customers arriving, phone calls coming in

The cumulative distribution function (cdf) is

$$F(x) = P[X \leq x] = \int_0^x \lambda e^{-\lambda y}\, dy$$

$$= \left[-e^{-\lambda y} \right]_0^x = 1 - e^{-\lambda x}$$

$$P[X \geq x] = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} \, dx = ?$$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} \, dx = ?$$

Easy with moment generating function.

$$\phi(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} \, dx = \lambda \int_0^\infty e^{-(\lambda - t)x} \, dx \quad {}^{*)}$$

$$= \lambda \left[-\frac{1}{\lambda - t} e^{-(\lambda - t)x} \right]_0^\infty = \frac{\lambda}{\lambda - t}.$$

Derivatives are

$$\phi'(t) = \frac{\lambda}{(\lambda - t)^2} \Rightarrow \phi'(0) = \frac{1}{\lambda}$$

$$\phi''(t) = \frac{2\lambda}{(\lambda - t)^3} \Rightarrow \phi''(0) = \frac{2}{\lambda^2}$$

*) Note: This integral only exists for $\lambda - t > 0$,

that is, $t < \lambda$.

However, for the mgf to be relevant, it is enough if it is defined in some interval around 0.

$$Var(X) = \phi''(0) - (\phi'(0))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Waiting makes the next event <u>not</u> more probable:

$$P[X \geq s+t \mid X \geq s] = \frac{P[X \geq s+t, X \geq s]}{P[X \geq s]}$$

$$= \frac{P[X \geq s+t]}{P[X \geq s]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda s - \lambda t} e^{\lambda s} = e^{-\lambda t} = P[X \geq t]$$

That is: the probability that we have to wait at least t is the same if we have waited already for s.

<u>Example</u> <u>59</u> Three machines work for an exponentially distributed period, with breakdown rate λ.
We use two until one breaks. That one is replaced by the unused one (say U).
What is the probability that the next machine to break is U?

Answer: The breakdown rate for both machines is the same. The fact that one machine has functioned already for a while has no effect on a possible breakdown (exp. distribution). Therefore, the probability is $\frac{1}{2}$.

Proposition 60: Suppose $X_1, \dots, X_n$ are independent RVs, with rates $\lambda_1, \dots, \lambda_n$. Then

$$X := \min(X_1, \dots, X_n)$$

is exponentiall distributed with rate

$$\lambda = \lambda_1 + \dots + \lambda_n .$$

Proof: $P[X \geq x] = P[X_1 \geq x, \dots, X_n \geq x]$

$= P[X_1 \geq x] \cdots P[X_n \geq x] = e^{\lambda_1 x} \cdot \dots \cdot e^{\lambda_n x}$

$= e^{(\lambda_1 + \dots + \lambda_n) x}$

Usage: Determine when the first component of a system will fail

3.2 The Poisson Distribution

Let $X_1, X_2, \dots$ be independent exponentially distributed RVs with rate λ. We interpret the X_i as consecutive waiting times:

- X_1 is the time until the first event happens
- X_2 is the subsequent time until the second event happens

etc.

What is the probability that exactly k events happen during the interval $[0,1]$ (e.g., within one hour, one day, etc.)?

One can show (see PDF document on OLE) that this is

$$\frac{\lambda^k}{k!} e^{-\lambda}$$

A discrete RV X is Poisson-distributed if

$$P[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \dots$$

The PMF of such an X is

$$\phi(t) = E[e^{tX}] = e^{-\lambda} \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

$$\phi'(t) = e^{\lambda(e^t-1)} \lambda e^t$$

$$\phi''(t) = e^{\lambda(e^t-1)} \lambda e^t \lambda e^t + e^{\lambda(e^t-1)} \lambda e^t$$

$$= \lambda e^t e^{\lambda(e^t-1)} (\lambda e^t + 1)$$

$\Rightarrow$

$$\mu = \phi'(0) = \lambda$$

$$\phi''(0) = \lambda^2 + \lambda \Rightarrow \sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

The Poisson distribution provides a good approximation for binomial RVs with parameters (n, p) if n is large and p is small.

Let X be such a RV, $X \sim Binom(n, p)$.

Let $\lambda := np$.

Then

$$P[X=k] = \frac{n!}{(n-k)!\,k!}\, p^k (1-p)^{n-k}$$

$$= \frac{n!}{(n-k)!\,k!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k}\,\frac{\lambda^k}{k!}\,\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Recall that $\lim\limits_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$. Then:

$$\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}$$

$$\left(1-\frac{\lambda}{n}\right)^i \approx 1$$

$$\frac{n(n-1)\cdots(n-k+1)}{n^k} \approx 1$$

Therefore

$$P[X=k] \approx \frac{\lambda^k}{k!} e^{-\lambda}$$

Example 56: Assume, on average there are 3 accidents on the highway between Trento and BZ. What is the probability there is at least one accident this week?

Answer: The number of accidents is Poisson-distributed (many cars, low probability of accident for single car) with $\lambda = 3$.

$$P[X \geq 1] = 1 - P[X < 1] = 1 - P[X=0]$$

$$= 1 - e^{-3}\frac{3^0}{0!} = 1 - e^{-3} \approx 0.95$$

Poisson RVs are reproductive.

Let X_1, X_2 be indep. Poisson RVs with rates λ_1, λ_2.
What about $X := X_1 + X_2$?

$$E[e^{tX}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} e^{tX_2}]$$
$$= E[e^{tX_1}] E[e^{tX_2}]$$
$$= e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)}$$
$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

This is the mgf of an exp. distributed RV with rate $\lambda_1 + \lambda_2$.

Example 58. The number of customers in a bar is on average 4 per hour. What is the probability that there are no more than 3?

Answer: Let X_1 be the number of customers in the first hour, X_2 in the second.
- X_1, X_2 are independent
- $X_1 + X_2$ is Poisson with $\lambda = 4 + 4 = 8$

Hence

$$P[X_1 + X \le 3] = \sum_{k=0}^{3} e^{-8} \frac{8^k}{k!} = 0.423$$

We can compute <u>mean</u> and <u>variance</u> of Poisson RVs also without mgfs.

Let $X \sim \text{Poisson}(\lambda)$. Then

$$E[X] = e^{-\lambda} \sum_{k=0}^{\infty} k \, P[X=k]$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \lambda e^{\lambda} = \lambda$$

$$E[X^2] = e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{d}{d\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= e^{-\lambda} \lambda \frac{d}{d\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \frac{d}{d\lambda} (\lambda e^{\lambda}) = e^{-\lambda} \lambda (e^{\lambda} + \lambda e^{\lambda})$$

$$= \lambda + \lambda^2$$

Hence,

$$E[X] = \lambda \quad , \qquad \text{Var}(X) = \lambda$$

We check that Poisson RVs are <u>reproductive</u> without mgfs.

Let $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$, X_1, X_2 indep.

$$Y := X_1 + X_2$$

Then

$$\begin{aligned}
P[Y = k] &= \sum_{j=0}^{k} P[X_1 = j, X_2 = k-j] \\
&= \sum_{j=0}^{k} P[X_1 = j] \cdot P[X_2 = k-j] \\
&= \sum_{j=0}^{k} e^{-\lambda_1} \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \frac{\lambda_2^{k-j}}{(k-j)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{j=0}^{k} \frac{1}{k!} \frac{k!}{j!(k-j)!} \lambda_1^j \lambda_2^{k-j} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda_1^j \lambda_2^{k-j} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^k}{k!}
\end{aligned}$$

That is, $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$