3 Special Random Variables

3.1 Bernoulli and Binomial

Experiment with two outcomes: success or failure

$$X = \begin{cases} 1 & \text{if success} \\ 0 & \text{if failure} \end{cases}$$

$P[X=1] = p$ for some $0 \leq p \leq 1$
(the probability of success)

X is a <u>Bernoulli</u>-RV

Let us repeat (independently!) the experiment n times and let X_n count the number of successes. Then X_n is a <u>binomial</u> RV with parameters (n,p). We have

$$P[X=i] = \binom{n}{i} p^i (1-p)^{n-i}$$

Example: A satellite system has 4 components and functions if at least 2 are working. Each component is independently working with probability $p = 0.6$.
What is the probability the system functions?

Solution: $P[\text{System functions}]$

$$= 1 - P[\text{System doesn't function}]$$

$$= 1 - p_0$$

$$p_0 = \binom{4}{0}(1-p)^4 + \binom{4}{1} p(1-p)^3$$

$$= 0.4^4 + 4 \cdot 0.6 \cdot 0.4^3$$

$$= 0.1792$$

Let $X \sim Bern(p)$ (X is a Bernoulli w/ parameter p)

$$E[X] = p$$

$$E[X^2] = E[X] = p$$

$$Var(X) = p - p^2 = p(1-p)$$

Let $X \sim B(n,p)$ (X is a binomial w/ parameters n, p)

$$E[X] = np$$

$$Var(X) = np(1-p)$$

<u>Computation of Probabilities for $B(n,p)$</u>:

$$P[X = k+1] = \frac{n!}{(k+1)!(n-k-1)!} p^{k+1} (1-p)^{(n-k-1)}$$

$$P[X = k] = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$\Rightarrow P[X = k+1] = P[X = k] \frac{n-k}{k+1} \cdot \frac{p}{1-p}$$

3.3 Uniform Random Variables

The continuous RV X is uniformly distributed over $[\alpha, \beta]$ if its density is

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$$

If $\alpha \leq a < b \leq \beta$, then

$$P[a \leq X \leq b] = \int_a^b f(x)\, dx$$

$$= \frac{b-a}{\beta - \alpha}$$

Also:

$$E[X] = \int_\alpha^\beta x \frac{1}{\beta-\alpha}\, dx = \frac{1}{\beta-\alpha} \left[\frac{x^2}{2} \right]_\alpha^\beta$$

$$= \frac{1}{\beta-\alpha} \frac{1}{2}(\beta^2 - \alpha^2) = \frac{(\beta+\alpha)(\beta-\alpha)}{2(\beta-\alpha)}$$

$$= \frac{\alpha+\beta}{2}$$

$$E[X^2] = \int_\alpha^\beta x^2 \frac{1}{\beta-\alpha} dx = \frac{1}{\beta-\alpha} \left[\frac{x^3}{3} \right]_\alpha^\beta$$

$$= \frac{1}{3} \frac{\beta^3 - \alpha^3}{\beta-\alpha} = \frac{1}{3}(\beta^2 + \alpha\beta + \alpha^2)$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \frac{(\alpha+\beta)^2}{2^2}$$

$$= \frac{4\beta^2 + 4\alpha\beta + 4\alpha^2 - 3\alpha^2 - 6\alpha\beta - 3\beta^2}{12}$$

$$= \frac{1}{12}(\beta^2 - 2\alpha\beta + \alpha^2) = \frac{(\beta-\alpha)^2}{12}$$

Values of a uniform RV on $[0,1]$ are called <u>random numbers</u>.

Suppose X_1, X_2 have a uniform joint distribution on the rectangle $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$. Then

$$f(x,y) = \begin{cases} \frac{1}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)} & \text{if } \alpha_1 \le x \le \beta_1, \\ & \quad \alpha_2 \le y \le \beta_2 \\ 0 & \text{otherwise} \end{cases}$$

Clearly:

- f_{X_1} is uniform on $[\alpha_1, \beta_1]$
- f_{X_2} is uniform on $[\alpha_2, \beta_2]$
- $f(x,y) = f_{X_1}(x) \cdot f_{X_2}(y)$

That is, X_1, X_2 are independent