2.8 The Weak Law of Large Numbers

We derive some inequalities to estimate probabilities in terms of mean and variance.

Let $X \geq 0$ s.th. $E[X]$ exists. Let also $a > 0$. Then

$$E[X] = \int_0^\infty x\, f(x)\, dx \geq \int_a^\infty x\, f(x)\, dx$$

$$\geq \int_a^\infty a\, f(x)\, dx = a \int_a^\infty f(x)\, dx = a\, P[X \geq a]$$

Markov's Inequality:

$$P[X \geq a] \leq \frac{E[X]}{a}$$

Apply Markov's Inequality to

$$Y := (X-\mu)^2 \quad , \quad a = k^2$$

Assume that $Var(X) = E[Y] = \sigma^2$ exists. Then

$$\frac{\sigma^2}{k^2} = \frac{E(Y)}{k^2} \geq P[Y \geq k^2] = P[(X-\mu)^2 \geq k^2]$$

$$= P[\,|X-\mu| \geq k\,]$$

Chebyshev's Inequality:

$$P[\,|X-\mu| \geq k\,] \leq \frac{\sigma^2}{k^2}$$

<u>Example 52</u> Suppose working time of a person is a RV X with $\mu = 40$ hrs

1) How probable is it, the person will work more than 60 hrs?

2) If $Var(X) = 16$, how probable is it, the person will work between 32 and 48 hrs?

Solution

1) Markov's inequality:

$$P[X \geq 60] \leq \frac{\mu}{60} = \frac{40}{60} = \frac{2}{3}$$

2) Chebyshev's inequality:

$$P[\,|X-\mu| \geq 8\,] \leq \frac{\sigma^2}{8^2} = \frac{16}{64} = \frac{1}{4}$$

Hence

$$P[\,32 \leq X \leq 48\,] \geq \frac{3}{4}$$

What happens if we execute an experiment many times and take averages of the outcomes?

Let X be a RV. Let $X_1, \dots, X_n$ be RVs that

1) have the same distribution as X
2) are independent.

Let $\sigma^2 = Var(X) = Var(X_i)$, $\mu = E[X] = E[X_i]$

Let $Y := \frac{\sum_{i=1}^n X_i}{n}$.

$$\Rightarrow E[Y] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} n \cdot \mu = \mu$$

$$\Rightarrow Var(Y) = Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right)$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i\right) \underset{\text{independence}}{=} \frac{1}{n^2} \sum_{i=1}^n Var(X_i)$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

Apply Chebyshev's inequality to Y and $k := \varepsilon > 0$:

$$P\left[\left|\frac{\sum_{i=1}^n X_i}{n} - \mu\right| > \varepsilon\right] = P[|Y - \mu| > \varepsilon]$$

$$\leq \frac{\frac{\sigma^2}{n}}{\varepsilon^2} = \frac{1}{n} \frac{\sigma^2}{\varepsilon^2} \xrightarrow[n \to \infty]{} 0$$

Theorem (Weak Law of Large Numbers)

Let $X_1, \dots, X_n, \dots$ be independent identically distributed RVs with $E[X_i] = \mu$. Then for every $\varepsilon > 0$

$$\lim_{n \to \infty} P\left[\left|\frac{1}{n} \sum_{i=1}^n X - \mu\right| > \varepsilon\right] = 0$$