## 3.4 Exponential Random Variables

Let $X$ be a random variable that stands for the time we have to wait for a radioactive atom to decay (or for some other similar event).

We assume that the waiting time does not depend on the time we have already waited. To some extent this holds also when waiting for

- the next customer
- the next email
- the next taxi.

Let $X$ be the waiting time. Our assumption that the probability distribution for the time after $s$, if the event has not occured, is the same as the original distribution can be expressed as

$$(*) \quad P[X > s+t \mid X > s] = P[X > t]$$

let $F(t) := P[X \leq t]$ and $G(t) := P[X > t] = 1 - F(t)$.

Then

$$\lim_{t \to \infty} G(t) = 0$$

because $G(t) = 1 - F(t)$ and $\lim_{t \to \infty} F(t) = 1$.

The definition of conditional probabilities tells us that (*) is equivalent to

$$\frac{P[X > s+t]}{P[X>s]} = \frac{P[X > s+t, \; X > s]}{P[X>s]}$$

$$= P[X > s+t \mid X > s] = P[X > t]$$

That is,

$$G(s+t) = P[X > s+t]$$

$$= P[X>s] \cdot P[X>t] = G(s) \cdot G(t)$$

This yields

$$G(t) = a^t \quad \text{where} \quad a = G(1)$$

and $a < 1$ since $\lim_{t \to \infty} G(t) = 0$.

Since $a < 1$, we have $\log a < 0$.

Let $\lambda := -\log a$. Then $a^t = e^{(\log a)t} = e^{-\lambda t}$.

Hence, $G(t) = P[X > t] = e^{-\lambda t}$.

$$\Rightarrow \quad F(t) = P[X \leq t] = 1 - P[X > t] = 1 - G(t)$$
$$= 1 - e^{-\lambda t}$$

is the cdf (= distribution function) of $X$

$$\Rightarrow \quad f(t) = \frac{d}{dt} 1 - e^{-\lambda t} = \lambda e^{-\lambda t}$$ is the pdf of $X$.

We say $X$ is exponentially distributed with parameter $\lambda$, written

$$X \sim Exp(\lambda)$$

What does $\lambda$ stand for? The dimension of $t$ is time.

$\Rightarrow$ The dimension of $\lambda$ is $\text{time}^{-1}$,

i.e., $\lambda$ is a frequency or rate.

In the labs we have calculated:

(using integration by parts)

- $E[X] = \int_0^\infty t \cdot e^{-\lambda t}\, dt = \frac{1}{\lambda}$

$\frac{1}{\lambda}$ is the average waiting time

$$E[X^2] = \int_0^\infty t^2 \cdot e^{-\lambda t}\, dt = \frac{2}{\lambda^2}$$

$\lambda$ is the average number of events per time unit, i.e., the rate of events

- $Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Hence, $\mu = \frac{1}{\lambda}$, $\sigma = \frac{1}{\lambda}$

# Multiple Mailboxes

We assume that the arrival of E-mails can be modeled by an exponential distribution. That is, there is a rate $\lambda > 0$ such that the probability to wait at least for a time $t$ for the next mail is

$$e^{-\lambda t} = G(t)$$

Suppose there are $n$ people with an E-mail mailbox and the rate at which mail arrives at mailbox $i$ is $\lambda_i$.

What is the probability that no message will arrive at any of the mailboxes during the next time period of $t$ if arrivals at different boxes are independent?

Let $X_i$ be the waiting time for a message to arrive at mailbox $i$.

Then $X_i \sim Exp(\lambda_i)$.

The probability that no mail arrives at box $i$ during time $t$ is

$$P[X_i > t] = e^{-\lambda_i t} \qquad = G_i(t)$$

The probability that no mail arrives at any box is

$$
\begin{aligned}
P[X_1 > t \,\&\, \ldots \,\&\, X_n > t] \\
&= P[X_1 > t] \cdot P[X_2 > t] \cdot \ldots \cdot P[X_n > t] \qquad \text{independence of } X_i \\
&= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot \ldots \cdot e^{-\lambda_n t} \qquad G_1(t) \cdots G_n(t) \\
&= e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n) t}
\end{aligned}
$$

$$\boxed{\begin{array}{l}\underline{\text{Proposition 60}}: \text{ If } X_1, \ldots, X_n \text{ are independent RVs, } X_i \sim Exp(\lambda_i), \\ \text{then} \\ \qquad \min(X_1, \ldots, X_n) \sim Exp(\lambda_1 + \cdots + \lambda_n)\end{array}}$$

## 3.2 The Poisson Distribution

The Poisson distribution models a scenario where a sequence of events happens:

- the time between events is distributed exponentially with rate $\lambda$
- the times between two events are independent of each other.

We are then interested in how many events happen during an interval of unit length (the length to which the rate $\lambda$ refers.

The Poisson distribution gives us the probability that exactly $k$ events happen during a unit interval.

To apply it, we need the rate $\lambda$ and we have to verify that the underlying assumptions hold.

**Examples:** Essentially the same as for the exponential distribution. Question answered can be about the number of

- customers arriving per hour at a shop during a workday afternoon
- emails arriving per minute at a mailserver
- soldiers being killed by horse kicks per year (classical application in Germany around 1900)

Whether assumption are satisfied can be checked by

- measuring the average waiting time $T$
- checking whether the times are $\text{Exp}(\frac{1}{T})$ distributed.

Let $X_1, X_2, \dots$ be independent exponentially distributed RVs with rate $\lambda$. We interpret the $X_i$ as consecutive waiting times:

- $X_1$ is the time until the first event happens
- $X_2$ is the subsequent time until the second event happens

etc.

What is the probability that exactly $k$ events happen during the interval $[0,1]$

(e.g., within one hour, one day etc.)?

This problem deals with the sum of i.i.d $Exp(\lambda)$ RVs.

Given the $X_i$, let

$$S_k := \sum_{i=1}^{k} X_i .$$

sum of waiting times for first $k$ events

and let

$$N := \arg\max_k \left( S_k \leq 1 \right),$$

maximal $k$ s.th $k$ events happen in one time unit

that is, $N$ is the maximal number of consecutive $X_i$, starting with $i=1$, whose sum does not exceed 1.

Note, $N$ is discrete. What is

$$P[N=k],$$

$k = 0, \ldots, k, \ldots$ ?

Probability of exactly $k$ events in a unit time

## Proposition:

$$N = k \iff S_k \leq 1 \text{ and } X_{k+1} \geq 1 - S_k$$

Plan: Let $f$ be the density of $X_{k+1}$ and $f_k$ of $S_k$.

Then

- $f(s) \cdot f_k(t)$ is the joint distribution of $X_{k+1}$ and $S_k$

- $P[N=k] = P[S_k \leq 1, X_{k+1} > 1 - S_k]$

$$= \boxed{\int_0^1 f_k(t) \int_{1-t}^{\infty} f(s) \, ds \, dt}$$

Note:

- $S_k \geq 0$
- $X_k \geq 0$
- $S_k, X_{k+1}$ indep

We know $f(s) = \lambda e^{-s}$.

But what is $f_k$?

General consideration: Let $\mathcal{X}, \mathcal{Y}$ be independent,

$\mathcal{X} \sim f(x)$, $\mathcal{Y} \sim g(y)$.

Then

$$\mathcal{X} + \mathcal{Y} \sim f * g$$

where

$$\boxed{(f * g)(z) = \int_{-\infty}^{\infty} f(x) \cdot g(z-x) \, dx}$$

- $f * g$ is the convolution of $f$ and $g$

- Iterate over all combinations of numbers that sum up to $t$:

  $x + (z-x) = z$

- Multiply their probabilities:

  $f(x) \cdot g(z-x)$

- Sum them up:

  integrate

Find out $f_k$!

- $f_1(t) = \lambda e^{-\lambda t}$

- $f_2(t) = (f_1 * f_1)(t) = \int_0^t f_1(s) f_1(t-s)\, ds$

$$= \int_0^t \lambda e^{-\lambda s} \cdot \lambda e^{-\lambda(t-s)}\, ds$$

$$= \lambda^2 \int_0^t e^{-\lambda(s+t-s)}\, ds$$

$$= \lambda^2 \int_0^t e^{-\lambda t}\, ds = \lambda^2 e^{-\lambda t} \int_0^t ds$$

$$= \lambda^2 t e^{-\lambda t}$$

- $f_3(t) = (f_2 * f_1)(t)$

$$= \int_0^t \lambda^2 s e^{-\lambda s} \lambda e^{-\lambda(t-s)} \, ds$$

$$= \lambda^3 e^{-\lambda t} \int_0^t s \, ds = \lambda^3 \frac{t^2}{2} \cdot e^{-\lambda t}$$

- $f_k(t) = \lambda^k \frac{t^{k-1}}{(k-1)!} e^{-\lambda t} \sim \mathcal{S}_k$

Density of k-fold sum of i.i.d exponential RV.s with frequency $\lambda$

Also called Gamma distribution $\Gamma(k, \frac{1}{\lambda})$

Now, let's calculate:

$$P[N=k] = P[\overbrace{\mathcal{S}_k \le 1}^{\sim f_k}, \overbrace{\mathcal{X}_{k+1} > 1 - \mathcal{S}_k}^{\sim f}]$$

$$= \int_0^1 f_k(t) \left( \int_{1-t}^{\infty} f(s)\, ds \right) dt$$

$$= \int_0^1 \lambda^k \frac{t^{k-1}}{(k-1)!} e^{-\lambda t} \int_{1-t}^{\infty} \lambda e^{-\lambda s}\, ds\, dt$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^1 t^{k-1} e^{-\lambda t} \left[ -e^{-\lambda s} \right]_{1-t}^{\infty} dt$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^1 t^{k-1} e^{-\lambda t} e^{-\lambda(1-t)}\, dt$$

We continue:

$$\boxed{P[N=k]} = \frac{\lambda^k}{(k-1)!} \int_0^1 t^{k-1} e^{-\lambda t} e^{-\lambda(1-t)} \, dt$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^1 t^{k-1} e^{-\lambda} \, dt$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^1 t^{k-1} \, dt \; e^{-\lambda}$$

$$= \frac{\lambda^k}{(k-1)!} \left[ \frac{t^k}{k} \right]_0^1 e^{-\lambda} = \boxed{\frac{\lambda^k}{k!} e^{-\lambda}}$$

This is the pmf of the Poisson distribution

with rate $\lambda$, $\mathrm{Pois}(\lambda)$

$\lambda = 3$

Example 56: Assume, on average there are three accidents per week on the highway between Trento and BZ. What is the probability that there is at least one accident this week?

Three accidents per week $\Longrightarrow$ frequency $\lambda = 3$

$\mathcal{A}$ = # accidents $\sim$ Pois$(3)$

In general: $P[m \le \mathcal{A} \le n] = \sum_{k=m}^{n} P[\mathcal{A} = k] = \frac{\lambda^k}{k!} e^{-\lambda}$

Here: $P[\mathcal{A} \ge 1] = 1 - P[\mathcal{A} < 0]$

$$= 1 - P[\mathcal{A} = k]$$

$$= 1 - \frac{3^0}{0!} e^{-3} = 1 - e^{-3}$$

Probability of at least 5 accidents per week.

$$P[A \geq 5] = \sum_{k=5}^{\infty} \frac{3^k}{k!} e^{-3}$$

$$= 1 - \sum_{k=0}^{4} \frac{3^k}{k!} e^{-3}$$

Probability of at least 5 accidents in two weeks:

- new unit time: 2 weeks instead of 1
- new frequency: 6 per two weeks
- new RV $A_2$ (= # accidents in 2 weeks) $\sim \mathrm{Pois}(3+3) = \mathrm{Pois}(6)$

$$\Longrightarrow P[A_2 \geq 5] = 1 - \sum_{k=0}^{4} \frac{6^k}{k!} e^{-6}$$

# Mean and variance

$\text{Exp}(\lambda)$ has rate $\lambda$,

i.e., $\lambda$ events per time unit

$\Rightarrow$ $\text{Pois}(\lambda)$ has mean $\lambda$ ?

Proof: Let $X \sim \text{Pois}(\lambda)$. Then

$$E[X] = \sum_{k=\cancel{0}\,1}^{\infty} \cancel{k} \frac{\lambda^k}{\underset{(k-1)!}{\cancel{k}!}} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}$$

$$= \lambda e^{\lambda} \cdot e^{-\lambda} = \lambda$$

$$E[X^2] = \sum_{k=\cancel{0}\,1}^{\infty} k^{\cancel{2}} \frac{\lambda^k}{\underset{(k-1)!}{\cancel{k}!}} e^{-\lambda} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} (k+1) \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \left( \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} 1 \cdot \frac{\lambda^k}{k!} \right)$$

$$= \lambda e^{-\lambda} \left( \lambda e^{\lambda} + e^{\lambda} \right) = \lambda^2 + \lambda$$

$$Var(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

So, $\mu = \lambda$ and $\sigma^2 = \lambda$

# Poisson and Binomial

Mean of $B(n,p) = n \cdot p$

Mean of $Exp(\lambda) = \lambda$

View $\lambda = n \cdot p$ *)

$\Rightarrow p = \frac{\lambda}{n}$

Suppose $X \sim B(n,p)$

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

*) Idea: Probability $p$ (small!!) for a car to have an accident.

Many cars, $n$ (large!!).

$\Rightarrow$ Rate of accidents $= n \cdot p = \lambda$.

$$P[X=k] = \frac{n!}{k!\,(n-k)!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$= \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{\approx 1} \frac{\lambda^k}{k!} \frac{\overbrace{\left(1-\frac{\lambda}{n}\right)^n}^{\to e^{-\lambda}}}{\underbrace{\left(1-\frac{\lambda}{n}\right)^k}_{\to 1}}$$

$$\lim_{n\to\infty}\frac{\lambda}{n} = 0$$

Therefore, $P[X=k] \approx \frac{\lambda^k}{k!} e^{-\lambda}$

or

$$B(n,p) \approx \text{Pois}(n\cdot p)$$

for large $n$ and small $p$.

Note: This is a rule of thumb from the time when computers were rare and slow.

**Example 58** The number of customers in a bar is on average 4 per hour. What is the probability that there are no more than 3 in 2 hours?

Let $X_1$ = # customers in $1^{st}$ hour

$X_2$ = # customers in $2^{nd}$ hour

Remember the Poisson story

- $X_1, X_2$ independent $\Rightarrow$ $X_1 + X_2$ Poisson
- Rate of $X_1 + X_2$ is $4+4=8$ in 2 hours

Reproductive property of the Poisson

$$P[X_1 + X_2 \leq 3] = \sum_{k=0}^{3} e^{-8} \frac{8^k}{k!} = 0.423$$