Exponential Functions

Three concepts: We consider functions with the following properties:

1) $f(x) = a^x$, $x \in \mathbb{Q}$ (i.e., x like $\frac{4}{4}$, $\frac{13}{9}$, ...), for some $a > 0$

Exponentiation

a^1, a^2, $a^{1/4} = \sqrt[4]{a}$, $a^0 = 1$

$1 = a^{-3} \cdot a^3 = a^0 \implies a^{-3} = \frac{1}{a^3}$

$a^{-\frac{7}{3}} = \frac{1}{\sqrt[3]{a^7}}$

2) $f(x+y) = f(x) \cdot f(y)$, $x, y \in \mathbb{R}$

Addition-multiplication homomorphism

3) $f'(x) = \alpha f(x)$, $x \in \mathbb{R}$ and $f(0) = 1$, for some $\alpha \neq 0$.

Growth proportional to value

like

$(e^{\alpha x})' = \alpha e^{\alpha x}$

We will see that these three concepts are actually equivalent for differentiable functions.

If a differentiable function f has one of these three properties, then it also has the other two.

We show that

1) implies 2)

2) implies 3)

3) implies 2)

2) implies 1)

We note that exponentiation can also be defined for real numbers as exponents. This, however, is only conceptually interesting, it does not give us a practical way to compute such powers. That will come later.

If $x \in \mathbb{R}$ is a real number, then we can approximate it by rational numbers. That is, there is a sequence r_n such that

$$\lim_{n \to \infty} r_n = x, \quad \text{or} \quad r_n \to x.$$

Then we define

$$a^x := \lim_{n \to \infty} a^{r_n}$$

For instance, this gives

$$5^\pi = \lim \left(5^3, 5^{\frac{31}{10}}, 5^{\frac{314}{100}}, \dots \right)$$

Implication 1) $\Rightarrow$ 2)

If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and for some $a \in \mathbb{R}$

$$f(x) = a^x, \quad x \in \mathbb{Q}$$

Exponentiation

then

$$f(x+y) = f(x) \cdot f(y), \quad x, y \in \mathbb{R}$$

Addition - multiplication homomorphism

Proof: By the properties of exponentiation, we have

$$f(x+y) = f(x) \cdot f(y) \quad \text{for all } x, y \in \mathbb{Q}.$$

If f is differentiable, it is also continuous and the second equation holds also for $x, y \in \mathbb{R}$ because addition and multiplication are continuous.

Implication 2) $\Rightarrow$ 3)

If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f(x+y) = f(x) \cdot f(y), \quad x, y \in \mathbb{R}$$

then there is a constant $\alpha \in \mathbb{R}$ such that

$$f'(x) = \alpha f(x), \quad x \in \mathbb{R} \qquad \text{and} \qquad f(0) = 1.$$

Proof: First, we note that $f(0) = 1$.

This is because

$$f(0) = f(0+0) = f(0) \cdot f(0)$$

$$\Rightarrow \quad 1 = f(0)$$

Next, we see what we can conclude about f':

$$\boxed{f'(x)} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x) \cdot 1}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(h) - 1}{h} \cdot f(x) \right)$$

$$= \left(\lim_{h \to 0} \frac{f(h) - f(0)}{h} \right) \cdot f(x)$$

$$= \boxed{f'(0) \cdot f(x)}$$

So, $f'(0)$ is the α we were looking for.

Implication 3) $\Rightarrow$ 2)

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and there is a constant $\alpha \in \mathbb{R}$ such that

$$f'(x) = \alpha f(x), \quad x \in \mathbb{R} \qquad \text{and} \qquad f(0) = 1$$

then

$$f(x+y) = f(x) \cdot f(y), \quad x, y \in \mathbb{R}$$

Proof: This argument is a bit lengthy. We first check that it is enough to prove the claim for $\alpha = 1$. From there we arrive at the power series of the exponential function and obtain the homomorphism equation.

First: It's enough to consider $\alpha = 1$.

Suppose that $g'(x) = \alpha\, g(x)$ and $g(0) = 1$.

We normalize g as f, defined as $f(x) := g\left(\frac{1}{\alpha} x\right)$.

We get back g from f because $g(x) = g\left(\alpha \cdot \frac{1}{\alpha} x\right) = f(\alpha x)$.

Then $f'(x) \overset{\text{chain rule}}{=} g'\left(\frac{1}{\alpha} x\right) \cdot \frac{1}{\alpha} \overset{\text{proportional growth}}{=} \cancel{\alpha}\, g\left(\frac{1}{\alpha} x\right) \cdot \cancel{\frac{1}{\alpha}} = g\left(\frac{1}{\alpha} x\right) = f(x)$.

That is, f satisfies $f' = f$.

We also have $f(0) = g\left(\frac{1}{\alpha} 0\right) = g(0) = 1$.

Suppose we can show that such an f also satisfies

$$f(x+y) = f(x) \cdot f(y).$$

Then $g(x+y) = f(\alpha(x+y)) = f(\alpha x + \alpha y) = f(\alpha x) \cdot f(\alpha y) = g(x) \cdot g(y)$.

So, it suffices to consider $\alpha = 1$.

Second: What does f look like?

It cannot be a polynomial like $f(x) = a_0 + a_1 x^1 + \dots + a_n x^n$.

This would yield $f^{(n+1)} = 0$. $f^{(n+1)}$ is the $(n+1)$-th derivative

Let us assume that f has a power series, i.e., f is an infinitely long polynomial $f(x) = a_0 + a_1 x^1 + \dots + a_n x^n + \dots$.

What does that tell us about the coefficients a_n?

$$f(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + a_{n+1} x^{n+1}$$

$$f'(x) = 0 + a_1 x^0 + 2 a_2 x^1 + 3 a_3 x^2 + \dots + n \cdot a_n x^{n-1} + (n+1) a_{n+1} x^n$$

The series f' and f are the same iff they have the same coefficients: $a_1 = a_0$, $2 a_2 = a_1$, $\dots$, $(n+1) a_{n+1} = a_n$.

That is, they satisfy the recurrence

$$a_{n+1} = \frac{1}{n+1} a_n \quad \text{with} \quad a_0 = 1.$$

The recurrence

$$a_{n+1} = \frac{a_n}{n+1} \qquad \text{with} \qquad a_0 = 1$$

leads to the values

$$a_0 = 1, \qquad a_1 = \frac{1}{1}, \qquad a_2 = \frac{1}{1\cdot 2}, \qquad a_3 = \frac{1}{1\cdot 2\cdot 3}$$

and generally

$$a_n = \frac{1}{n!}.$$

The shape of f is therefore

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This function is also known as the exponential function and it is often denoted as exp.

We see, its form derives from the two conditions

$$f' = f \qquad \text{and} \qquad f(0) = 1.$$

Third: The exponential function satisfies $f(x+y) = f(x) \cdot f(y)$.

We start with the right-hand side:

$$f(x) \cdot f(y) = \left(\sum_{i=0}^{\infty} \frac{x^i}{i!} \right) \cdot \left(\sum_{j=0}^{\infty} \frac{y^j}{j!} \right)$$

Reorganize the sum, combine factors whose exponents add up to n, for each n.

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}$$

Multiply the inner sum by $1 = \frac{n!}{n!}$.

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} x^k y^{n-k}$$

Binomial formula!

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = f(x+y)$$

Implication 2) $\Rightarrow$ 1)

If

$$f(x+y) = f(x)\cdot f(y), \quad x,y \in \mathbb{R}$$

then for some $a \in \mathbb{R}$

$$f(x) = a^x, \qquad x \in \mathbb{Q}$$

Proof: We have already shown that from our assumption it follows that

- $f(0) = 1.$

We conclude from $1 = f(0) = f(x+(-x)) = f(x)\cdot f(-x)$ that

- $f(-x) = \dfrac{1}{f(x)}.$

Moreover,

- $$f(m\cdot x) = f(\underbrace{x+\cdots+x}_{m \text{ times}}) = \underbrace{f(x)\cdot\cdots\cdot f(x)}_{m \text{ times}} = f(x)^m$$

From $$f(x) = f(\underbrace{\tfrac{x}{n}+\tfrac{x}{n}+\cdots+\tfrac{x}{n}}_{n \text{ times}}) = f\left(\tfrac{x}{n}\right)^n$$

we conclude

- $$f\left(\tfrac{x}{n}\right) = \sqrt[n]{f(x)} = f(x)^{1/n}$$

Hence, for every rational number $\frac{m}{n}$ we have

- $$f\left(\tfrac{m}{n}\right) = f\left(m\cdot\tfrac{1}{n}\right) = f\left(\tfrac{1}{n}\right)^m = \left(f(1)^{1/n}\right)^m = f(1)^{m/n}$$

So far we have seen that $f(x+y) = f(x) \cdot f(y)$

implies $$f(x) = f(0)^x, \quad x \in \mathbb{Q}$$

For the special case of $f = \exp$, that is, $f' = f$,

we have $$f(1) = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We often denote the number $\exp(1)$ simply as e.

Then we have $$e^x = \exp(1)^x = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{Q}$$

Since $\exp$ is differentiable, (this was always our assumption)

it is continuous on $\mathbb{Q}$, this equality also holds for $x \in \mathbb{R}$.

If g satisfies $g'(x) = \alpha g(x)$, then $g(x) = \exp(\alpha x)$, as seen before, that is,

$$g(x) = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{n!}.$$

We have $\exp(x) > 0$ for $x > 0$ and $\exp(-x) = \frac{1}{\exp(x)}$, hence $\exp(x) > 0$ holds also for $x < 0$.

That is, $\exp'(x) = \exp(x) > 0$ for all $x \in \mathbb{R}$.

Thus, exp is strictly monotonic and has an inverse function that we call log.

As the inverse of exp, the function log inherits the property

$$\log(x \cdot y) = \log(x) + \log(y).$$

The known laws for logarithms and exponents can all be derived from the development shown so far.