PTS - Chapter 3

3 Special Random Variables

3.1 Bernoulli and Binomial

A Bernoulli experiment has only 2 outcomes

$$X = \begin{cases} 1 & \text{if success} \\ 0 & \text{if failure} \end{cases}$$

Pmf of X:

$$P[X=1] = p, \qquad \text{for some } 0 \le p \le 1$$

$$P[X=0] = (1-p)$$

Let X be a Bernoulli variable with probability p. Then

$$\mathbb{E}[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

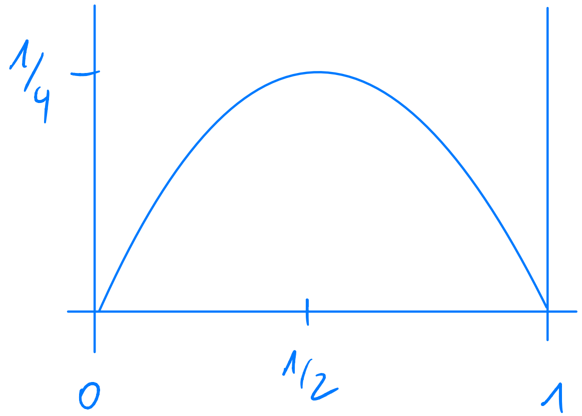
Note: $X^2 = X$

$$\mathbb{E}[X^2] = \mathbb{E}[X] = p$$

$$\operatorname{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= p - p^2 = p(1-p)$$

maximal for $p = \frac{1}{2}$



Graph of $p(1-p)$,

maximal for $p = \frac{1}{2}$

Let us repeat a Bernoulli experiment with RV X.

Suppose $X_1, \dots, X_n, \dots$ are i.i.d. $\mathrm{Bern}(p)$ RVs.

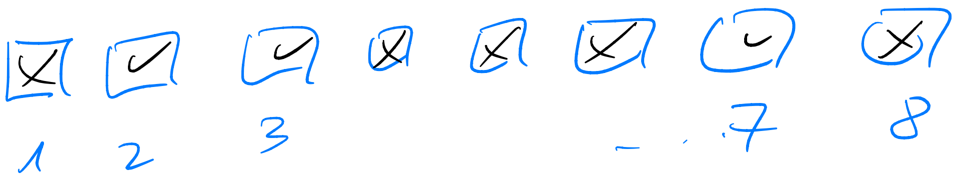
Then

$$Y_n := \sum_{i=1}^{n} X_n$$ counts the number of successes

Possible values of Y_n: $0, 1, \dots, n$

$$P[Y_n = i] = \binom{n}{i} p^i (1-p)^{n-i}$$

Boxes to record success



3 successes

- # ways to get 3 successes? $\binom{8}{3}$
- Probability of such an outcome: $p^3 (1-p)^{8-3}$

Let us repeat a Bernoulli experiment with RU X n times

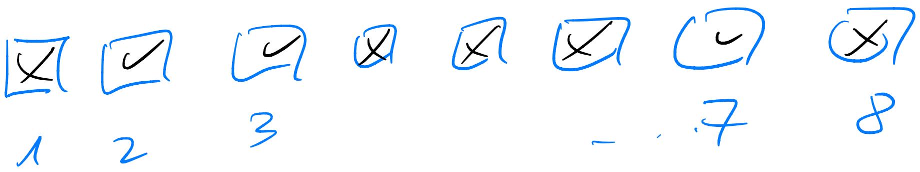
Suppose $X_1, \dots, X_n$ are i.i.d. Bern(p) RUs.

Then
$$Y_n := \sum_{i=1}^{n} X_n$$
counts the number of successes

The possible values of Y_n are $0, 1, 2, \dots, n$.

$$P[Y_n = i] = \binom{n}{i} p^i (1-p)^{n-i}$$

Boxes to record success



3 successes

- # ways to get 3 successes? $\binom{8}{3}$

- Probability of such an outcome: $p^3 (1-p)^{8-3}$

Note:

$$(p+q)^n = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i}$$

according to the binomial theorem.

Suppose $q = 1-p$. Then

$$1 = (p + (1-p))^n = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

This shows that

$$p_i := \binom{n}{i} p^i (1-p)^{n-i}, \quad 0 \leq i \leq n,$$

is a probability mass function.

We say that Y_n is distributed according to the binomial distribution with parameters n and p, written

$$Y_n \sim B(n, p).$$

We calculate mean and variance:

$$E[Y_n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

$$Var(Y_n) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

$$= \sum_{i=1}^{n} p(1-p) = np(1-p)$$

Example: A satellite system has 4 components and functions if at least 2 are working.

Each component is independently working with probability $p = 0.6$.

What is the probability that the system functions?

$P[\text{System functions}]$

$= 1 - P[\text{System doesn't function}]$

$= 1 - p_0$

$p_0 = P[\text{all components fail}] + P[\text{exactly 3 components fail}]$

$= P[\text{no component functions}] + P[\text{exactly 1 comp. functions}]$

$$= \binom{4}{0} p^0 (1-p)^4 + \binom{4}{1} p^1 (1-p)^3$$

$$= 1 \cdot 0.4^4 + 4 \cdot 0.6 \cdot 0.4^3 = 0.1792$$

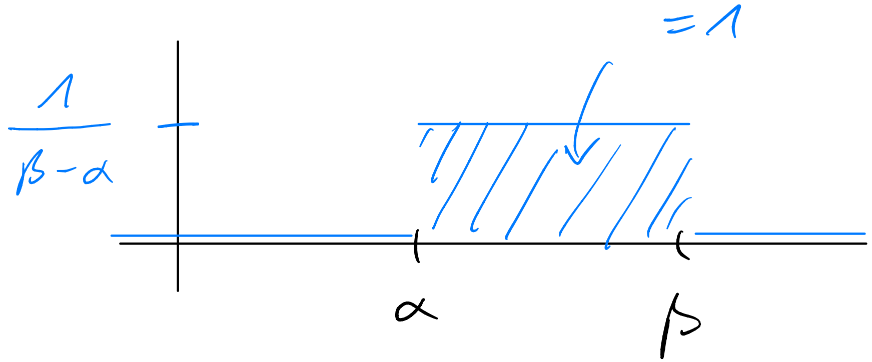
3.3 Uniform Random Variables

A continuous RV X is uniformly distributed if there is an interval $[\alpha, \beta]$ so that

- X takes only values in $[\alpha, \beta]$
- all values are equally probable.

This means, X has the density f where

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \alpha \le x \le \beta \\ 0 & \text{otherwise}. \end{cases}$$



We write $X \sim U[\alpha, \beta]$.

What about mean and variance?

We first determine mean and variance for the simple case that $X \sim U[0,1]$. Then

$$E[X] = \int_0^1 x \cdot 1 \, dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2 \cdot 1 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

Hence,

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{4}$$

Suppose now that $\mathcal{Y} \sim \mathcal{U}[\alpha, \beta]$.

Then $\mathcal{X} := \dfrac{\mathcal{Y} - \alpha}{\beta - \alpha}$ is $\mathcal{U}[0,1]$-distributed and

$$\mathcal{Y} = (\beta - \alpha)\mathcal{X} + \alpha.$$

Therefore,

$$E[\mathcal{Y}] = (\beta - \alpha) E[\mathcal{X}] + \alpha = \frac{\beta - \alpha}{2} + \alpha$$

$$= \frac{\alpha + \beta}{2}$$

$$\mathrm{Var}(\mathcal{Y}) = (\beta - \alpha)^2 \mathrm{Var}(\mathcal{X}) = \frac{(\beta - \alpha)^2}{12}$$

Mean of $U[\alpha,\beta]$: Brute-Force Calculation

This is a calculation of mean and variance following directly the definition. Compare this to our approach of (i) solving a simple variant of the problem and (ii) reducing complex cases to the simple one.

Also:

$$E[X] = \int_\alpha^\beta x \frac{1}{\beta-\alpha}\,dx = \frac{1}{\beta-\alpha}\left[\frac{x^2}{2}\right]_\alpha^\beta$$

$$= \frac{1}{\beta-\alpha}\frac{1}{2}(\beta^2-\alpha^2) = \frac{(\beta+\alpha)(\beta-\alpha)}{2(\beta-\alpha)}$$

$$= \frac{\alpha+\beta}{2}$$

Variance of $U[\alpha,\beta]$: Brute-Force Calculation

$$E[X^2] = \int_\alpha^\beta x^2 \frac{1}{\beta-\alpha}\,dx = \frac{1}{\beta-\alpha}\left[\frac{x^3}{3}\right]_\alpha^\beta$$

$$= \frac{1}{3}\,\frac{\beta^3-\alpha^3}{\beta-\alpha} = \frac{1}{3}\left(\beta^2+\alpha\beta+\alpha^2\right)$$

$$Var(X) = E[X^2]-E[X]^2$$

$$= \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \frac{(\alpha+\beta)^2}{2^2}$$

$$= \frac{4\beta^2+4\alpha\beta+4\alpha^2-3\alpha^2-6\alpha\beta-3\beta^2}{12}$$

$$= \frac{1}{12}\left(\beta^2-2\alpha\beta+\alpha^2\right) = \frac{(\beta-\alpha)^2}{12}$$