Joint distribution of RVs X, Y: Revision

Individual RVs can be described by

pmfs (discrete)

pdfs (continuous)

Joint pmfs: $p(x_i, y_j)$ if x_i, y_j are the possible values of X, Y

Joint pdfs: $f(x,y)$

$$\sum_{i,j} p(x_i, y_j) = 1$$

$$=$$

$$\sum_i \sum_j p(x_i, y_j)$$

$$=$$

$$\sum_j \sum_i p(x_i, y_j)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$$

$$=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

Marginal pmfs, pdfs

$$p_X(x_i) = \sum_j p(x_i, y_j) \qquad p_Y(y_j) = \sum_i p(x_i, y_j)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Independence of events

$\mathcal{E}, \mathcal{F}$ indep. if

- "inf about $\mathcal{E}$ does not provide information about $\mathcal{F}$"

- $P(\mathcal{F} \mid \mathcal{E}) = P(\mathcal{F})$ $\quad (\Leftrightarrow P(\mathcal{F} \mid \mathcal{E}) = P(\mathcal{E})$

$\Leftrightarrow P(\mathcal{E}\mathcal{F}) = P(\mathcal{E})P(\mathcal{F})$

Independence of RVs

X, Y are indep. if

all events that can be described in terms of X

"$5 < X < 9$", "$X > 2$", "$X < 0$ or $X > 2$"

are independent of events that can be described in terms of Y.

"$Y \geq 4$"

Proposition X, Y are independent iff

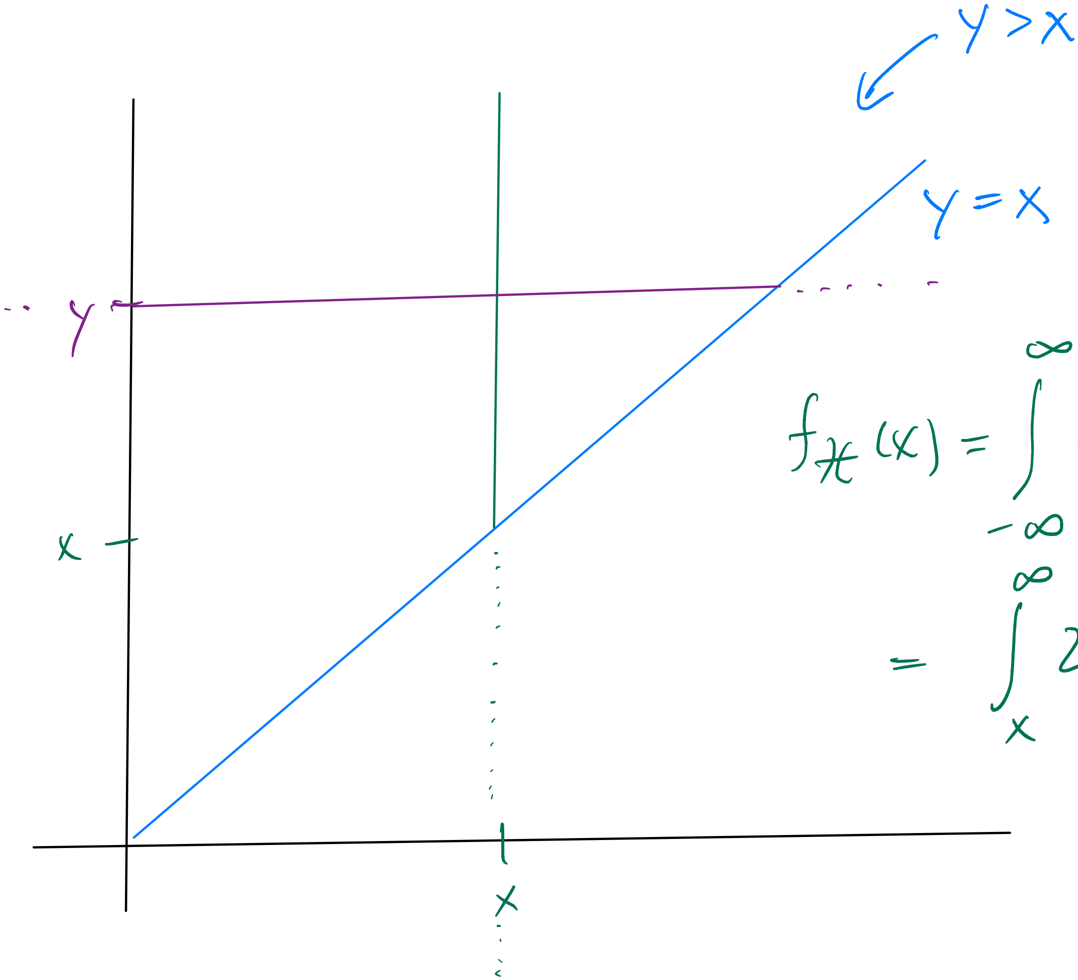
- $p(x_i, y_j) = p_X(x_i) \cdot p_Y(y_j)$, f.a. x_i, y_j

- $f(x,y) = f_X(x) \cdot f_Y(y)$, f.a. x, y

Marginal Densities and Independence

Suppose X, Y have the joint distribution

$$f(x,y) = \begin{cases} 2e^{-x}e^{-y} & 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^y 2e^{-x}e^{-y}\,dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_x^{\infty} 2e^{-x}e^{-y}\,dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{\infty} 2e^{-x}e^{-y}\,dy$$

$$= 2e^{-x} \int_{x}^{\infty} e^{-y}\,dy = 2e^{-x}\left[-e^{-y}\right]_{x}^{\infty}$$

$$= 2e^{-x}e^{-x} = 2e^{-2x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{y} 2e^{-x}e^{-y}\,dx$$

$$= 2e^{-y}\int_{0}^{y} e^{-x}\,dx = 2e^{-y}\left[-e^{-x}\right]_{0}^{y}$$

$$= 2e^{-y}\left(1 - e^{-y}\right)$$

Expected Values of RVs

Die: X has values $1, \dots, 6$, all with $p(x_i) = \frac{1}{6}$

$$E[X] = 1 \cdot p(1) + 2 \cdot p(2) + \cdots + 6 \cdot p(6)$$

$$= (1 + 2 + \cdots + 6) \frac{1}{6} = \frac{35}{6}$$

Discrete RV X

$$E[X] = \sum_i x_i \cdot p(x_i)$$

Continuous RV

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

2.5 Properties of Expectation

X RV, $g(x)$ function $\Longrightarrow$ $g(X)$ is a RV

X points of die, $g(x) = x^2$ $\Rightarrow$ $g(X) = X^2$, squares of points

X^2 is a new RV

values	probabilities
1	1/6
4	1/6
9	1/6
16	1/6
25	1/6
36	1/6

$$E[X^2] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + \cdots \cdots 36 \cdot \frac{1}{6}$$

$$= \frac{1+4+9+16+25+36}{6}$$

$$= \frac{91}{6}$$

Imagine: a die with numbers $-3, -2, -1, 1, 2, 3$

X is the number on top of the die: $E[X] = 0$

Let $Z := X^2$

1) Find the pmf of Z and compute $E[Z]$

$$E[Z] = 1 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 9 \cdot \frac{1}{3}$$

pmf of Z

values	probabilities
9	$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$
4	___ , ___
1	___ , ___

2) Take a weighted average of the $g(x_i)$

$$E[X^2] = (-3)^2 \cdot \frac{1}{6} + (-2)^2 \cdot \frac{1}{6} + (-1)^2 \cdot \frac{1}{6}$$

$$3^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{6}$$

$$= \sum_i x_i^2 \cdot p(x_i)$$

We found:

$$\boxed{E[g(X)] = \sum_i g(x_i) \cdot p(x_i)}$$

Proposition 39 ("Law of the Unconscious Statistician" LOTUS)

X RV, $g : \mathbb{R} \to \mathbb{R}$

- $E[g(X)] = \sum_i g(x_i)\, p(x_i)$
- $E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$

Applications of LOTUS

X RV, $a, b \in \mathbb{R}$, $g(x) = ax + b$

$$E[aX + b] \stackrel{?}{=} a\,E[X] + b$$

$$E[g(X)] = \int_{-\infty}^{\infty} (ax+b)\, f(x)\, dx = \int_{-\infty}^{\infty} ax\, f(x) + b f(x)\, dx$$

$$= a \int_{-\infty}^{\infty} x\, f(x)\, dx + \int_{-\infty}^{\infty} b \cdot 1\, f(x)\, dx$$

$$= a\,E[X] + b \cdot \int_{-\infty}^{\infty} 1\, f(x)\, dx$$

$$= a\,E[X] + b \cdot 1$$

Let X, Y be RVs. What about $E[X+Y]$?

$$E[X+Y] \stackrel{?}{=} E[X] + E[Y]$$ (Do we need independence?)

LOTUS also holds for 2-dim. densities:

Let X, Y be RVs with joint pdf $f(x,y)$, let $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

Then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y) \, dx \, dy$$

The summation law follows with $g(x,y) = x+y$

$E[g(X,Y)]$ $\qquad g(x,y)$

$$E[X+Y] \overset{\text{lotus}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} (x+y)\, f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} x \cdot f(x,y)\, dx\, dy + \int_{\mathbb{R}}\int_{\mathbb{R}} y \cdot f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}} x \underbrace{\left(\int_{\mathbb{R}} f(x,y)\, dy\right)}_{f_X(x)} dx + \int_{\mathbb{R}} y \underbrace{\left(\int_{\mathbb{R}} f(x,y)\, dx\right)}_{f_Y(y)} dy$$

$$= \int_{\mathbb{R}} x \cdot f_X(x)\, dx + \int_{\mathbb{R}} y \cdot f_Y(y)\, dy$$

$$= E[X] + E[Y]$$

Generalization

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Throwing 2 dice, adding result $\quad X_1 + X_2$

$$E[X_1 + X_2] = E[X_1] + E[X] = \frac{7}{2} + \frac{7}{2} = 7$$

Example 42: Tossing coin n times, $\quad E[\#\text{ heads}] = ?$

$$p(H) = p \, , \quad p(T) = 1-p$$

Let $X_i = 1$ iff head with i-th toss $\Rightarrow E[X_i] = p$

n times: $E\left[\sum_{i=1}^{n} X_i\right] = n \, E[X_1] = n p$

Note: $\quad E[X] = 1 \cdot p + 0 \cdot (1-p) = p$!!