2.4 Expectation

If throwing a die, we expect that over time the average of points is

$$\frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = 3.5$$

If tossing a coin n times in a row, we expect that, over many such sequences, the average number of heads is $\frac{n}{2}$.

If tossing a coin until we see the first head, we expect the sequence of tosses to have length

$$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \cdots$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k k = \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = 2$$

<u>Definition</u> Let X be a discrete RV with possible values $x_1, \ldots, x_n$. Then

$$E[X] := \sum_{i=1}^{n} x_i \, P[X = x_i]$$

is the <u>expected</u> <u>value</u> of X

<u>Definition</u> Let X be a continuous RV with density f. Then

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

is the <u>expected</u> <u>value</u> of X (if the integral exists)

<u>Example 36</u>: A fire alarm exercise is announced for Monday between 2pm and 4pm. The alarm can go off any time between 2 and 4 pm and all times are equally likely. How long do we expect to wait?

Waiting time X : $0 \le X \le 2$ hrs

Density $f(x) = \begin{cases} 1/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = \int_0^2 x \frac{1}{2}\, dx = \left[\frac{x^2}{4} \right]_0^2 = \frac{4}{4} = 1$$

2.5 Properties of Expectation

Example 37: We play a game with a die where a player gets k^2 points when throwing number k. What is the expected number of points? (X = number of die)

$$E[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \dots + \frac{1}{6} \cdot 6^2 = \frac{91}{6}$$

Example 38: Waiting time X is uniformly distributed over $[0,2]$ (see Ex. 36)
The cost of waiting for x hours ($x \in [0,2]$) is cx^3. What is the expected cost?

$$E[cX^3] = \int_0^2 cx^3 \frac{1}{2} dx = \left[c\frac{x^4}{4}\right]_0^4 = 4c.$$

Proposition: For a function $g: \mathbb{R} \to \mathbb{R}$,

$$E[g(X)] = \sum_{i=1}^{n} g(x_i)\, p(x_i)$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

Note: This needs a proof (although it is plausible)

X RV, $a, b \in \mathbb{R}$

$$E[aX+b] = \int_{-\infty}^{\infty} (ax+b)\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} a x\, f(x)\, dx + \int_{-\infty}^{\infty} b\, f(x)\, dx$$

$$= a \int_{-\infty}^{\infty} x f(x)\, dx + b \int_{-\infty}^{\infty} f(x)\, dx$$

$$= a\, E[X] + b$$

Holds also for discrete RVs

Let X, Y be RVs with joint density $f(x,y)$ and $g: \mathbb{R}^2 \to \mathbb{R}$. Then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\, dx\, dy$$

For $g(x,y) = x+y$, this yields

$$E[X+Y] = \int_{\mathbb{R}}\int_{\mathbb{R}} (x+y)\, f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} x\, f(x,y)\, dx\, dy + \int_{\mathbb{R}}\int_{\mathbb{R}} y\, f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}} x \int f(x,y)\, dy\, dx + \int_{\mathbb{R}} y \int_{\mathbb{R}} f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}} x\, f_X(x)\, dx + \int_{\mathbb{R}} y\, f_Y(y)\, dy$$

$$= E[X] + E[Y]$$

In general: $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$

Example 41: Expected value of the sum of two dice:

X_i = number of i-th die $\Rightarrow E[X_i] = \frac{7}{2}$

$\Rightarrow \quad E[X_1 + X_2] = E[X_1] + E[X_2] = 7$

Example 42: Expected number of heads after n coin tosses

$$X_i = \begin{cases} 1 & \text{if i-th throw is H} \\ 0 & \text{if i-th throw is T} \end{cases}$$

$E[X_i] = \frac{1}{2} \quad \Rightarrow E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \frac{n}{2}$