Probability Theory and Statistics **Fall-Winter 2020/21**

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Densities and Expected Values

Instructions: You have two weeks, until 30 November 2020, 23:55 h, to submit your solution to these assignments on OLE.
You can work out your solutions with a word processing system (Word, Latex) or by hand. It may be easier to write up your answers by hand because they are likely to contain symbolic calculations with fractions, powers, and integrals. If you submit handwritten solutions, make an effort to write clearly and to structure and comment your answers so that they are legible. If you write by hand, submit the answer as a scanned PDF document. (Don't submit a photo, but a scan. Photos are usually very hard to read.)
Each question has a weight, expressed in points. For your answer you get a mark on the scale from 0 to 30. The weight determines how much the mark contributes to the coursework mark and therefore to the final mark. Note that for the final mark, we consider the maximum of the exam mark and the mark for the question. Therefore, not much is lost if you get a low mark for one question: (i) it can be compensated by the exam mark, and (ii) it does not influence the marks for the other questions.
Your assignment should represent your own effort. However, you are not expected to work alone. It is fine to discuss the exercises and try to find solutions together, but each student shall write down and submit his/her solutions separately. It is good academic standard to acknowledge collaborators, so if you worked together with other students, please list their names.

1 Joint Density on a Rectangle

The joint probability density function of $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f(x,y) = c(x^2 + \frac{xy}{2}), \qquad 0 < x < 1, \quad 0 < y < 2.$$

1. Determine the constant c.
2. Compute the marginal density function of $\mathcal{X}$.
3. Compute the marginal density function of $\mathcal{Y}$.
4. Are $\mathcal{X}$ and $\mathcal{Y}$ independent?
5. Find $P[\mathcal{X} > \mathcal{Y}]$.

(Weight: 15% of this CW)

2 Uniform Density on a Triangle

The joint probability density function of $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f(x,y) = \begin{cases} 2 & 0 < x < y, \quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Compute the marginal density of $\mathcal{X}$
2. Compute the marginal density of $\mathcal{Y}$
3. Are $\mathcal{X}$ and $\mathcal{Y}$ independent?
4. Compute $P[\mathcal{X} > \mathcal{Y}]$?

(Weight: 15% of this CW)

3 Expected Value

The density function of $\mathcal{X}$ is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E(\mathcal{X}) = 3/5$, find a, b.

(Weight: 10% of this CW)

4 Duration of Games

Suppose that two teams, A and B, play a series of games that end when one of them has won n games. Suppose that each game played is, independently, won by team A with probability p. There are no draws.

1. Find the expected number of games that are played when $n = 2$.
2. For which p is this number maximal and for which is it minimal?

(Weight: 20% of this CW)

5 The True Opinion of Metereologists

Each night different meteorologists give us the "probability" that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\begin{array}{ll} 1 - (1-p)^2 & \text{if it does rain} \\ 1 - p^2 & \text{if it does not rain.} \end{array}$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of the weather.
Suppose now that a given meteorologist is aware of this and so wants to maximize his or her expected score. If this individual truly believes that it will rain tomorrow with

probability p^*, what value of p should he or she assert so as to maximize the expected score?

(Weight: 20% of this CW)

6 Maximum and Minimum of Uniformly Distributed Random Variables

Let $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ be independent random variables having the common density function

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{M}_{ax} = \max(\mathcal{X}_1, \ldots, \mathcal{X}_n)$.

1. Determine the distribution function F of $\mathcal{M}_{ax}$.

   Hint: The maximum of $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ is $\leq$ than a given number x, if all of $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ are $\leq x$.

2. What is the density f of $\mathcal{M}_{ax}$?

3. What is the expected value of $\mathcal{M}_{ax}$?

4. Answer the analogous question for the minimum $\mathcal{M}_{in}$.

(Weight: 20% of this CW)