$$P[X<a] = \int_0^a \int_0^\infty 2e^{-x}e^{-2y}\,dy\,dx$$
$$= \int_0^a 2e^{-x} \int_0^\infty e^{-2y}\,dy\,dx$$
$$= \int_0^a 2e^{-x} \left[-\frac{1}{2}e^{-2y}\right]_0^\infty dx$$
$$= \int_0^a e^{-x}(0+1) = \int_0^a e^{-x}\,dx = \left[-e^{-x}\right]_0^a = 1-e^{-a}$$

$$P[X<Y] = \int_0^\infty \int_x^\infty 2e^{-x}e^{-2y}\,dy\,dx$$
$$= \int_0^\infty e^{-x} \int_x^\infty 2e^{-2y}\,dy\,dx$$
$$= \int_0^\infty e^{-x}\left[-e^{-2y}\right]_x^\infty dx$$
$$= \int_0^\infty e^{-x}(0+e^{-2x})\,dx = \int_0^\infty e^{-3x}\,dx$$
$$= \left[-\frac{1}{3}e^{-3x}\right]_0^\infty = \frac{1}{3}$$

## 2.3 Independent Random Variables

$X, Y$ are independent iff

$$P[X \in A, Y \in B] = P[X \in A] \cdot P[Y \in B]$$

for all (reasonable) $A, B \subseteq \mathbb{R}$.

Equivalent (proof!)

$$P[X \leq a, Y \leq b] = P[X \leq a, Y \leq b] \quad , \text{f.a. } a, b \in \mathbb{R}$$

that is

$$F(a, b) = F_X(a) \cdot F_Y(b)$$

Equivalent for discrete R.Vs:

$$p(x, y) = p_X(x) \cdot p_Y(y) \qquad \text{f.a. } x, y \in \mathbb{R}$$

for the probability mass functions;

$$f(x, y) = f_X(x)\, f_Y(y) \qquad \text{f.a. } x, y \in \mathbb{R}$$

for the densities of continuous RVs.

Example 33. Let $X, Y$ be independent, each with density

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note: The probability $P[Y \leq 0] = 0$ because

$$P[Y > 0] = \int_0^\infty e^{-x}\,dx = 1.$$

What is the density of $\frac{X}{Y}$ ?

To get the distribution function, we calculate:

$$P[\mathcal{X}/\mathcal{Y} \leq a] = P[\mathcal{X} \leq a\mathcal{Y}]$$

$$= \int_0^\infty \int_0^{ay} e^{-x} e^{-y} \, dx \, dy$$

$$= \int_0^\infty e^{-y} \left[ -e^{-x} \right]_0^{ay} dy = \int_0^\infty e^{-y} \left( 1 - e^{-ay} \right) dy$$

$$= \int_0^\infty e^{-y} - \int_0^\infty e^{-(1+a)y} \, dy$$

$$= 1 - \left[ -\frac{1}{1+a} e^{-(1+a)y} \right]_0^\infty$$

$$= 1 + \left( 0 - \frac{1}{1+a} \right) = 1 - \frac{1}{1+a}$$

The density is

$$\frac{d}{da}\left(1 - \frac{1}{1+a}\right) = \frac{d}{da}\left(1 - (1+a)^{-1}\right) = (1+a)^{-2},$$

that is $f_{X/Y}(a) = \frac{1}{(1+a)^2}$

Joint probability distributions can be defined for RVs $X_1, \ldots, X_n$. Also

joint mass functions
joint densities
independence.

<u>Example 34</u> The price of a stock changes daily by amount $x$ with probability $p(x)$, with

$$p(x) = \begin{cases} 0.05 & x \in \{-3, 3\} \\ 0.10 & x \in \{-2, 2\} \\ 0.15 & x \in \{-1, 1\} \\ 0.40 & x = 0 \end{cases}$$

Let $X_i$ be the change on day $i$, and assume the $X_i$ are independent. Then

$$p[X_1 = -1, X_2 = 2, X_3 = 0] = p(1)p(2)p(0)$$
$$= 0.15 \times 0.10 \times 0.4$$
$$= 0.006$$