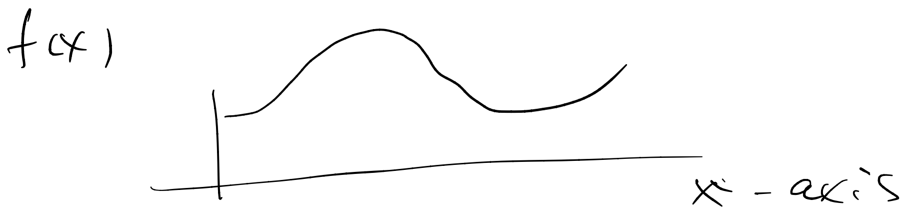
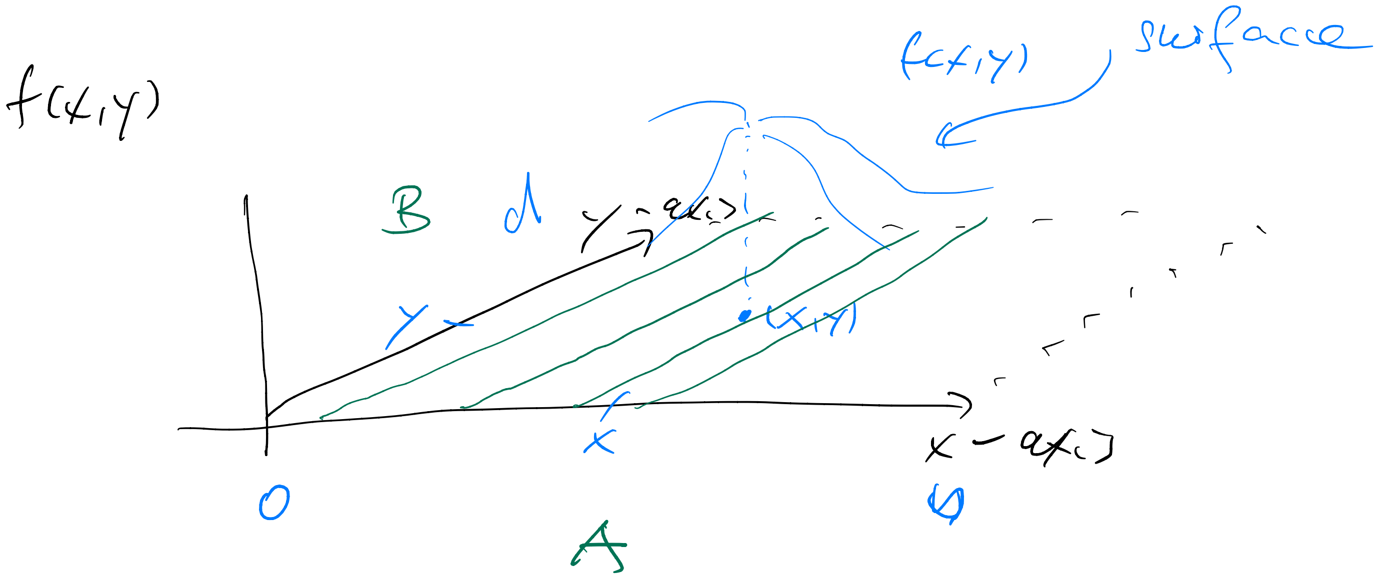
How can we model joint probabilities in the continuous case?

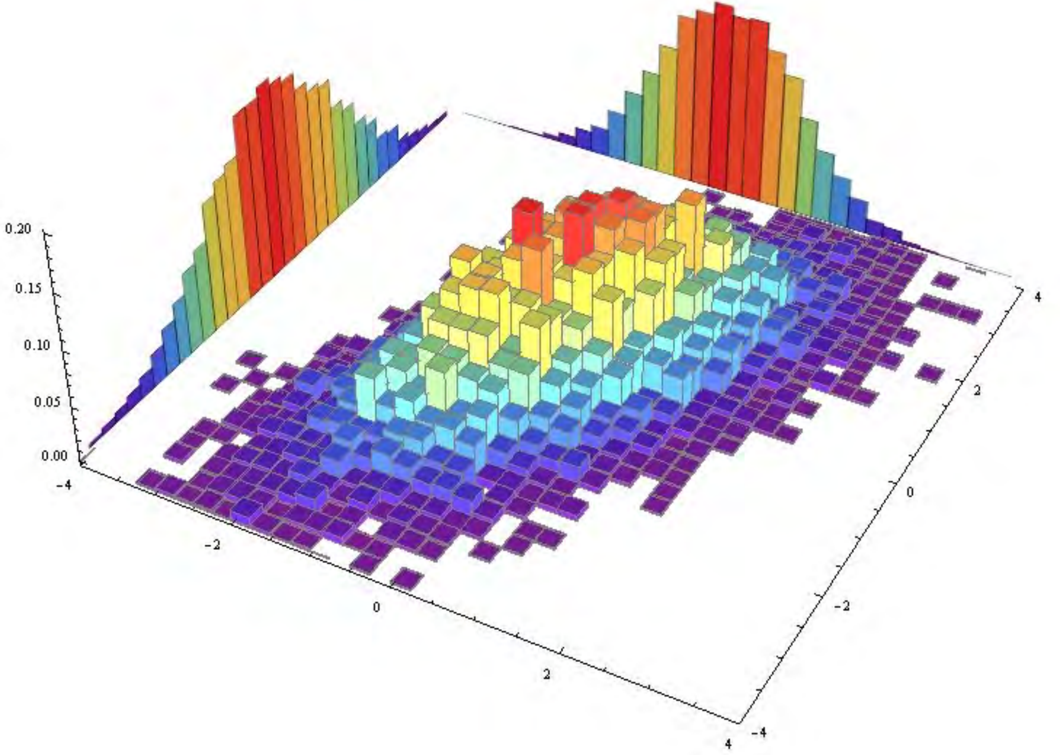
Discrete case: joint pmf $p(x,y)$ "discrete"

Continuous case: joint pdf $f(x,y)$





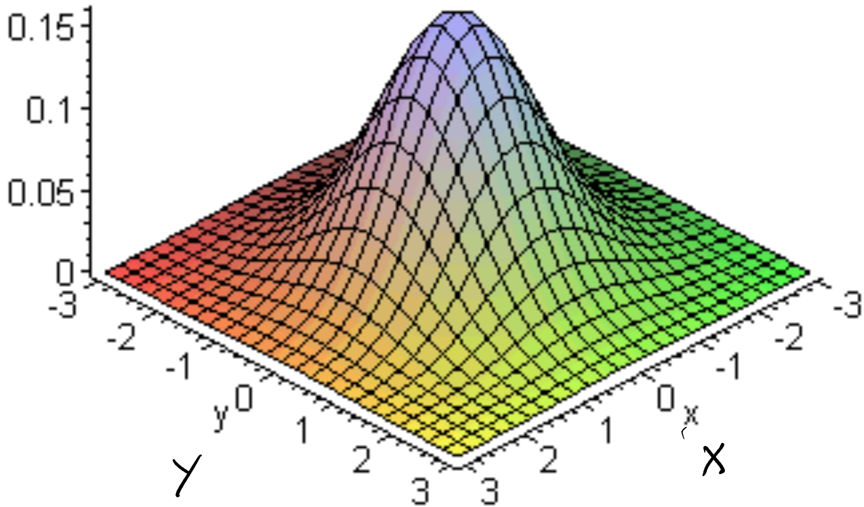
Integrals = volume under surface

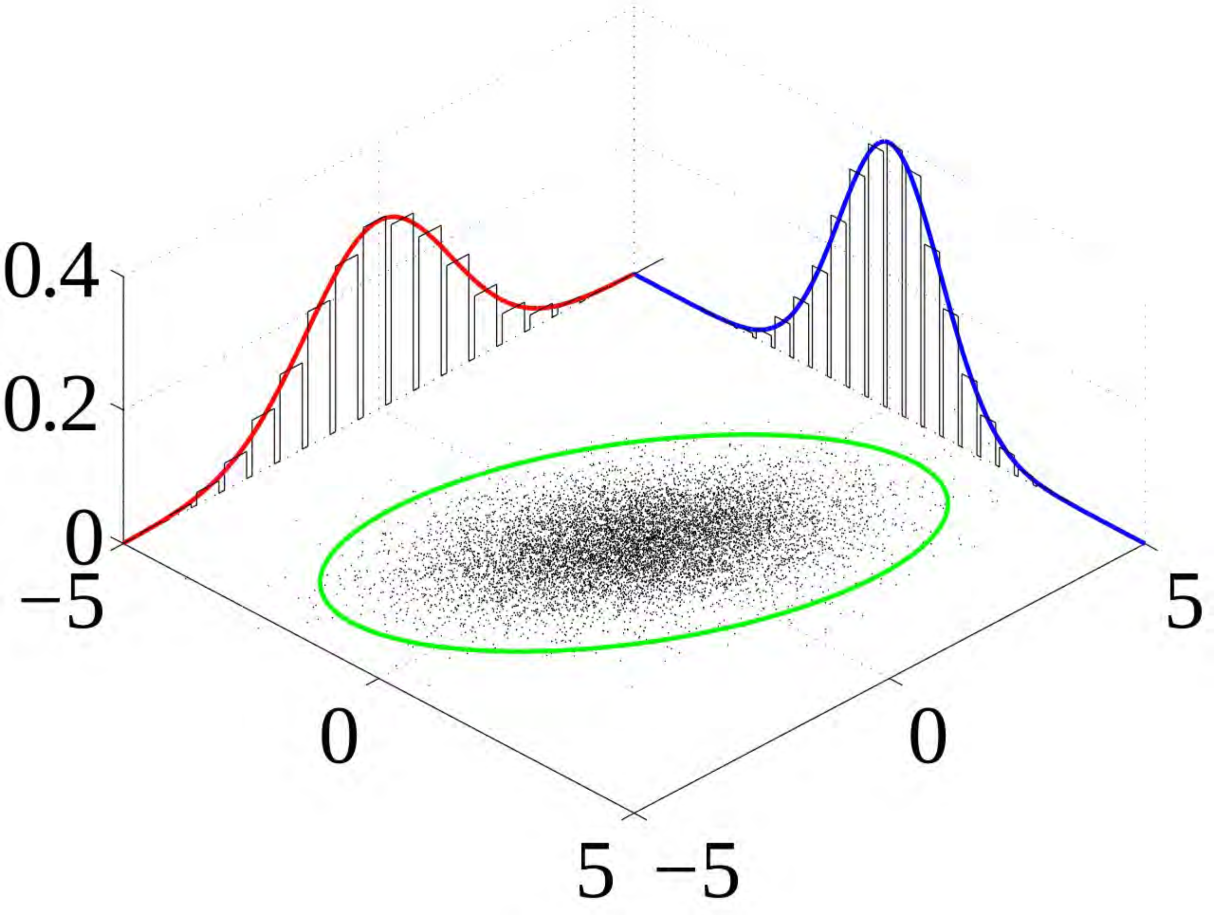


0.20
0.15
0.10
0.05
0.00
-4
-2
0
2
4
-4
-2
0
2
4

Bivariate Normal

Cavalieri's Principle




0.4
0.2
0
−5
0
5
−5
0
5

Let X, Y continuous, joint pdf is a fct

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R},$$

For $C \subseteq \mathbb{R} \times \mathbb{R}$, a reasonable subset, we have

$$P[(X, Y) \in C] = \iint_{(x,y) \in C} f(x,y)\, dx\, dy$$

Requirements of f:

$$f(x,y) \geq 0 \quad , \quad \iint_{\mathbb{R} \times \mathbb{R}} f(x,y)\, dx\, dy = 1$$

If $A, B \subseteq \mathbb{R}$, then

$$P[X \in A, Y \in B] = \int_A \left(\int_B f(x,y)\, dy \right) dx$$

$$= \int_B \left(\int_A f(x,y)\, dx \right) dy$$

Change of integration order is similar to change of summation order:

Consider

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$$

Then

$$\sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} = (a_{11} + a_{12}) + (a_{21} + a_{22})$$

$$\sum_{j=1}^{2} \sum_{i=1}^{2} a_{ij} = (a_{11} + a_{21}) + (a_{12} + a_{22})$$

The two sums are identical due to associativity and commutativity of addition.

Similarly, we have $\int_A \int_B f(x,y)\, dx\, dy = \int_B \int_A f(x,y)\, dy\, dx$

<u>Example 32:</u> Let the joint pdf of X, Y be

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

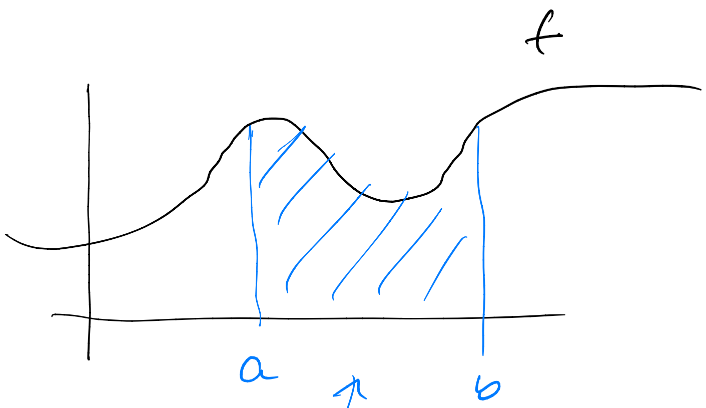
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{\infty} 2e^{-x}e^{-2y}\,dx\,dy$$

$$= \int_0^{\infty} 2e^{-2y}\left(\int_0^{\infty} e^{-x}\,dx\right)dy = \int_0^{\infty} 2e^{-2y}\left[-e^{-x}\right]_0^{\infty} dy$$

$$\int_0^{\infty} 2e^{-2y}(0-(-1))\,dy = \int_0^{\infty} 2e^{-2y}\,dy = \left[-e^{-2y}\right]_0^{\infty}$$

$$= 0 - (-e^{-2\cdot 0}) = 0 - (-1) = 1$$

Student Question: "When evaluating an integral of f with an antiderivative F, why do we plug the upper bound first into F?"



F is an antiderivative of f if

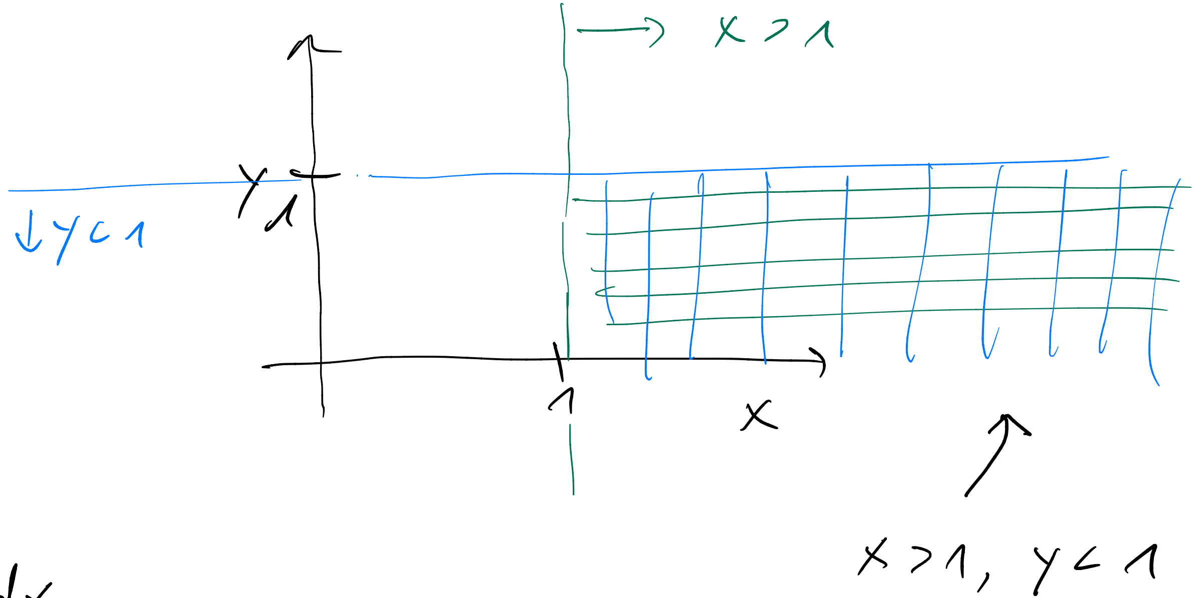
$F' = f$

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b$$

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx =$$

$$\lim_{b\to\infty} \left(F(b) - F(a)\right) = \left(\lim_{b\to\infty} F(b)\right) - F(a)$$

$$P[X > 1, y < 1]$$



$$= \int_1^\infty \int_0^1 f(x,y)\, dy\, dx$$

$$= \int_1^\infty \int_0^1 e^{-x}\, 2e^{-2y}\, dy\, dx$$

$$= \int_1^\infty e^{-x} \int_0^1 2e^{-2y}\, dy\, dx = \int_1^\infty e^{-x} \left[-e^{-2y}\right]_0^1 dx$$

$$= \int_1^\infty e^{-x} \left(-e^{-2} - (-e^{0})\right) dx = \int_1^\infty e^{-x}(1-e^{-2})\, dx$$

$$= (1-e^{-2}) \int_1^\infty e^{-x}\, dx = (1-e^{-2}) \left[-e^{-x}\right]_1^\infty$$

$$= (1-e^{-2})\, e^{-1} = e^{-1} - e^{-3}$$

$$\int_1^\infty \int_0^1 e^{-x} 2e^{-2y} \, dy \, dx$$

This is a constant that can be pulled out of the integral.

Special Case of:

$$\int_A \boxed{\int_B g(x) \cdot h(y) \, dy} \, dx = \int_A g(x) \boxed{\int_B h(y) \, dy} \, dx$$

$$= \int_B h(y) \, dy \cdot \int_A g(x) \, dx = \boxed{\int_A g(x) \, dx \cdot \int_B h(y) \, dy}$$

If

1) $f(x) = g(x) \, h(x)$

2) Integration area has form $A \times B$

Then

$$\iint_{A \times B} f \cdot g = \int_A f \cdot \int_B g$$

$$\int_1^\infty \int_0^1 e^{-x} 2e^{-2y} \, dy \, dx$$

$$= \int_1^\infty e^{-x} \, dx \cdot \int_0^1 2e^{-y} \, dy = \left[-e^{-x}\right]_1^\infty \cdot \left[-e^{-2y}\right]_0^1$$

$$= e^{-1}\left(1 - e^{-2}\right)$$

$$P[X < a] \qquad a > 0$$

$$= \int_0^a \int_0^\infty e^{-x} \cdot 2e^{-2y} \, dy \, dx$$

$$= \int_0^a e^{-x} \, dx \cdot \int_0^\infty 2e^{-2y} \, dy$$

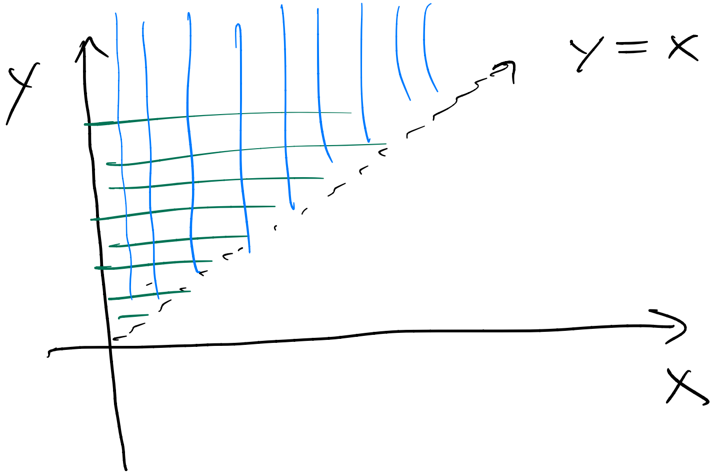
Density of Exp(2)

$$= \left[-e^{-x} \right]_0^a \cdot \left[-e^{-2y} \right]_0^\infty$$

$$= \left(e^0 - e^{-a} \right) \cdot \left(e^0 - 0 \right)$$

$$= \left(1 - e^{-a} \right) \cdot 1$$

$$P[X < Y]$$



$$= \int_0^\infty \int_x^\infty f(x,y)\, dy\, dx \qquad (*)$$

$$= \int_0^\infty \int_0^y f(x,y)\, dx\, dy$$

$$\overset{(*)}{=} \int_0^\infty \int_x^\infty e^{-x}\, 2e^{-2y}\, dy\, dx = \int_0^\infty e^{-x} \left[-e^{-2y} \right]_x^\infty dx$$

$$= \int_0^\infty e^{-x}\, e^{-2x}\, dx = \int_0^\infty e^{-3x}\, dx = \left[-\frac{1}{3} e^{-3x} \right]_0^\infty$$

$$= \frac{1}{3} e^{-3 \cdot 0} = \frac{1}{3}$$

2.3 Independent Random Variables

$\mathcal{E}, \mathcal{F}$ ind. $\iff P[\mathcal{E}\mathcal{F}] = P(\mathcal{E}) \cdot P(\mathcal{F})$

$\iff P(\mathcal{E} \mid \mathcal{F}) = P(\mathcal{E})$

($\mathcal{E} = $ "$X \in A$",

$\mathcal{F} = $ "$Y \in B$")

X, Y are independent iff

$$P[X \in A, Y \in B] = P[X \in A] \cdot P[Y \in B]$$

for all $A, B \subseteq \mathbb{R}$

Equivalent : $P[X \le a, Y \le b] = P[X \le a] \cdot P[Y \le b]$,

f.a. $a, b \in \mathbb{R}$

that is

$$F(a,b) = F_X(a) \cdot F_Y(b)$$

Equivalent for discrete RVs:

$$p(x,y) = p_X(x) \cdot p_Y(y) \qquad \text{f.a. } x,y \in \mathbb{R}$$

For cont. RVs

$$f(x,y) = f_X(x) \cdot f_Y(y) .$$

Example 33: Let X, Y be independent, each with density

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

What can we say about the quotient of two independent and exponentially distributed RVs?

What is the density of $\frac{X}{Y}$?

Two steps:

1) cdf of $\frac{X}{Y}$

2) pdf is derivative of cdf

1) $F(a) = P\left[X/Y \le a \right] = P\left[X \le aY \right]$

$$= \int_0^\infty \int_0^{ay} e^{-x} e^{-y} \, dx \, dy$$

$$= \int_0^\infty e^{-y} \left[-e^{-x} \right]_0^{ay} dy = \int_0^\infty e^{-y} \left(1 - e^{-ay} \right) dy$$

$$= \int_0^\infty e^{-y} dy - \int_0^\infty e^{-(1+a)y} \, dy$$

$$= 1 - \left[-\frac{1}{1+a} e^{-(1+a)y} \right]_0^\infty = 1 + \left[\cdots \right]_0^\infty$$

$$= 1 + \left(-\frac{1}{1+a} \right) = 1 - \frac{1}{1+a} \qquad \text{cdf}$$

2) $f(a) = \frac{d}{da} F(a) = - \frac{d}{da} (1+a)^{-1} = -(-1)(1+a)^{-2}$

$$= \frac{1}{(1+a)^2} \qquad \text{pdf}$$

Remark: Generalization to n RVs $X_1, \dots, X_n$ is possible:

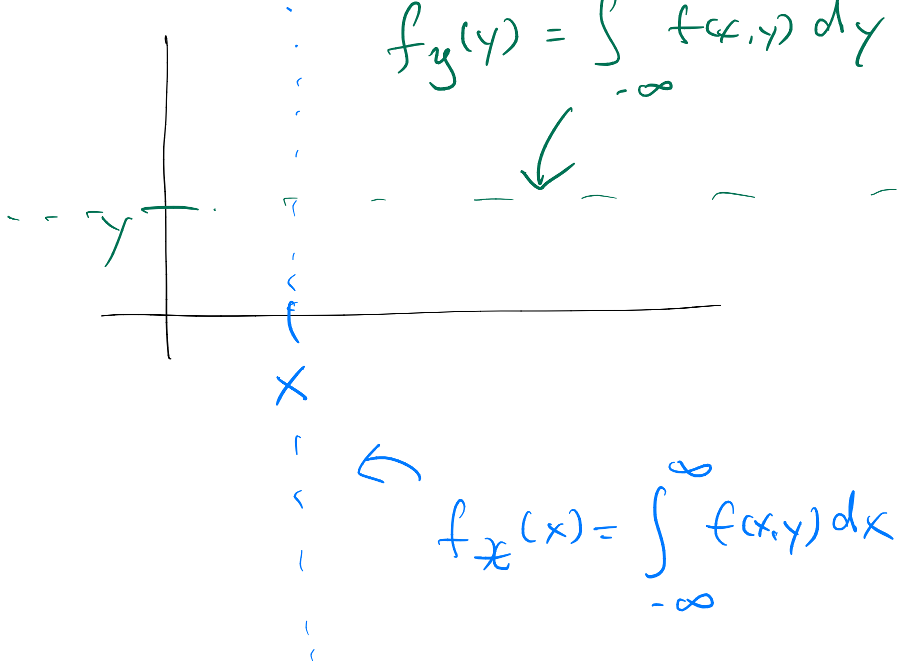
- joint pmf $\quad p(x_1, \dots, x_n)$
- joint pdf $\quad f(x_1, \dots, x_n)$
- independence

$$p(x_1, \dots, x_n) = p_{X_1}(x_1) \cdot \dots \cdot p_{X_n}(x_n)$$

$$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdot \dots \cdot f_{X_n}(x_n)$$

Marginal Densities

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \int_{-\infty}^{\infty} g(x)\,h(y)\,dy \\
&= g(x)\int_{-\infty}^{\infty} h(y)\,dy \\
&= g(x)\cdot 1 = g(x)
\end{aligned}
$$



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dx$$