We want to calculate:

$$\sum_{k=1}^{\infty} k \cdot \left(\frac{1}{2}\right)^k$$

Reminder: Geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x|<1$$

It turns out that a more general sum is easier to calculate:

$$\sum_{k=1}^{\infty} k \cdot x^k, \quad |x|<1$$

Note that

$$k \cdot x^{k-1} = \frac{d}{dx} x^k$$

Consequently,

$$k x^k = x \cdot k \cdot x^{k-1} = x \cdot \frac{d}{dx} x^k$$

Therefore, we can rewrite our sum as follows:

$$\sum_{k=1}^{\infty} k \cdot x^{k} = \sum_{k=1}^{\infty} x \cdot k \cdot x^{k-1} = x \cdot \sum_{k=1}^{\infty} k \cdot x^{k-1}$$

$$= x \cdot \sum_{k=1}^{\infty} \frac{d}{dx} x^{k} \overset{f'+g'=(f+g)'}{=} x \frac{d}{dx} \sum_{k=1}^{\infty} x^{k} \overset{\sum_{k=0}^{\infty} x^{k} = \frac{1}{1-x}}{=} x \cdot \frac{d}{dx} \frac{1}{1-x}$$

$$= x \cdot \frac{d}{dx} (1-x)^{-1} \overset{\frac{d}{dx} x^{\alpha} = \alpha x^{\alpha-1}}{=} x \cdot (-1) (1-x)^{-2} = \frac{x}{(1-x)^{2}}$$

Now, we can plug in $\frac{1}{2}$ for x:

$$\sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^{k} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^{2}} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^{2}} = \frac{1}{\frac{1}{2}} = 2$$

So, the expected number of coin tosses until the first head is 2.

Definition Let $\mathcal{X}$ be a discrete R.V., with values $x_1, \dots, x_k, \dots$

Then $$E[\mathcal{X}] := \sum_{i=1}^{n} x_i \, P[\mathcal{X} = x_i] \,, \quad \text{if } \mathcal{X} \text{ has } n \text{ values}$$

$$E[\mathcal{X}] := \sum_{i=1}^{\infty} x_i \, P[\mathcal{X} = x_i] \,, \quad \text{if } \mathcal{X} \text{ has } \infty \text{ many values,}$$

$E[\mathcal{X}]$ is the expected value of $\mathcal{X}$.

Definition Let $\mathcal{X}$ be a continuous R.V. with density f.

Then $$E[\mathcal{X}] := \int_{-\infty}^{\infty} x \, f(x) \, dx$$

is the expected value of $\mathcal{X}$

(if the integral exists)

$f(x) \cdot \Delta$

prob. of

$P[x \le \mathcal{X} \le x + \Delta]$

Waiting time for alarm :

$\mathcal{X}$ waiting time RV, takes value $[0,2]$

Density $f(x) = \begin{cases} 1/2 & x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$

$$E[\mathcal{X}] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^2 x \frac{1}{2}\, dx$$

$$= \left[\frac{1}{\cancel{2}} \frac{x^2}{2}\right]_0^2 = \left[\frac{x^2}{4}\right]_0^2 = \frac{2^2}{4} - \frac{0^2}{4} = \frac{4}{4} = 1$$

uniform probability

$c \cdot 2 = 1$

2.2 Joint Distributions

Consider two RVs X, Y together. Study probabilities

$$P[X = x, Y = y]$$

or

$$P[a < X \leq b, c < Y \leq d]$$

Example 29: 9 batteries, 2 new, 3 part. charged, 4 empty.

Randomly select 3 out 9 batteries.

X # new batteries $\qquad X \in \{0, 1, 2\}$

Y # partially charged $\qquad Y \in \{0, 1, 2, 3\}$

let $p(x,y) = P[X = x, Y = y]$ — joint pmf of X and Y

$$p(0,0) = \frac{\binom{4}{3}}{\binom{9}{3}} = \frac{4}{84} \qquad p(0,1) = \frac{\binom{3}{1}\binom{4}{2}}{\binom{9}{3}} = \frac{18}{84}$$

$$p(0,2) = \frac{\binom{3}{2}\binom{4}{1}}{\binom{9}{3}} = \frac{12}{84} \qquad p(0,3) = \frac{\binom{3}{3}}{\binom{9}{3}} = \frac{1}{84}$$

$$p(1,0) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{9}{3}} = \frac{12}{84}$$

$$\binom{9}{3} = \frac{\cancel{9}^{3} \cdot \cancel{8}^{4} \cdot 7}{\cancel{3} \cdot \cancel{2} \cdot 1} = 12 \cdot 7 = 84$$

$$p(1,1) = \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{\binom{9}{3}} = \frac{24}{84}$$

$$p(1,2) = \frac{\binom{2}{1}\binom{3}{2}}{\binom{9}{3}} = \frac{6}{84}$$

$$p(2,0) = \frac{\binom{2}{2}\binom{4}{1}}{\binom{9}{3}} = \frac{4}{84}$$

$$p(2,1) = \frac{\binom{2}{2}\binom{3}{1}}{\binom{9}{3}} = \frac{3}{84}$$

We have computed the joint pmf of X and Y.

Summarize in table (by multiples of $\frac{1}{84}$)

$X \backslash Y$	0	1	2	3	Sum
0	4	18	12	1	35
1	12	24	6	0	42
2	4	3	0	0	7
Sum	20	45	18	1	84

joint pmf

probability of $X = i$, i.e. $P[X = i]$

marginal probabilities

prob. of $Y = j$, i.e. $P[Y = j]$

Joint cumulative probability distrib.

$$F(x,y) = P[X \leq x, Y \leq y]$$

Distribution of X

$$F_X(x) = P[X \leq x] = P[X \leq x, Y < \infty]$$
$$= F(x, \infty)$$

How we get the marginal pmf of $\mathcal{X}$ out of the joint pmf $p(x,y) = P[\mathcal{X}=x, \mathcal{Y}=y]$?

$$p_{\mathcal{X}}(x) = P[\mathcal{X}=x] = \sum_{j=1}^{n} P[\mathcal{X}=x, \mathcal{Y}=y_j]$$

$$= \sum_{j=1}^{n} p(x, y_j)$$

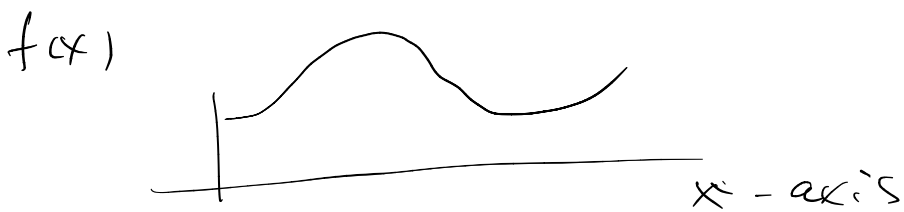
$$p_{\mathcal{Y}}(y) = P[\mathcal{Y}=y] = \sum_{i=1}^{m} p(x_i, y)$$

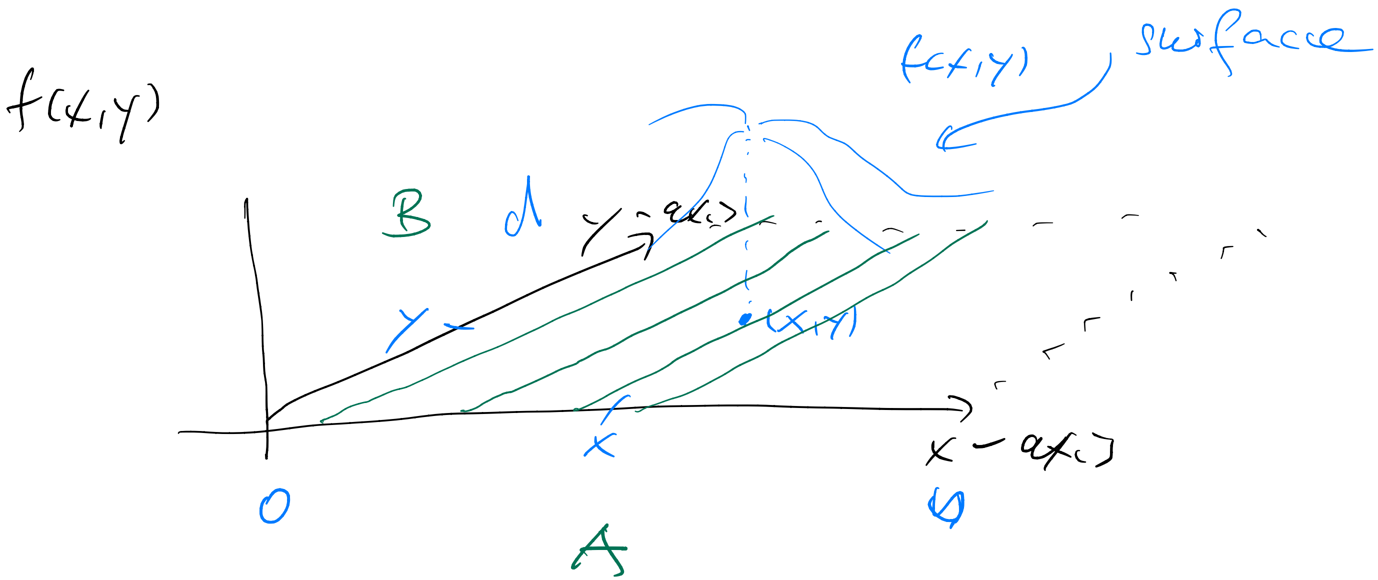
How can we model joint probabilities in cont. case?

discrete case: joint pmf $p(x,y)$

continuous case: joint pdf $f(x,y)$

(p. density f.)





Integrals = areas under surface

Let X, Y continuous, joint pdf is a fct

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R},$$

For $C \subseteq \mathbb{R} \times \mathbb{R}$, a reasonable subset, we have

$$P[(X, Y) \in C] = \iint_{(x,y) \in C} f(x,y) \, dx \, dy$$

Requirements of f:

$$f(x,y) \geq 0, \quad \iint_{\mathbb{R} \times \mathbb{R}} f(x,y) \, dx \, dy = 1$$

If $A, B \subseteq \mathbb{R}$, then

$$P[X \in A, Y \in B] = \int_A \left(\int_B f(x,y) \, dy \right) dx$$

$$= \int_B \left(\int_A f(x,y) \, dx \right) dy$$

<u>Example 32</u>: Let the joint pdf of X, Y be

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & x,y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{\infty} 2e^{-x}e^{-2y}\,dx\,dy$$

$$= \int_0^{\infty} 2e^{-2y}\left(\int_0^{\infty} e^{-x}\,dx\right)dy = \int_0^{\infty} 2e^{-2y}\left[-e^{-x}\right]_0^{\infty} dy$$

$$\int_0^{\infty} 2e^{-2y}(0-(-1))\,dy = \int_0^{\infty} 2e^{-2y}\,dy = \left[-e^{-2y}\right]_0^{\infty}$$

$$= 0-(-e^{-2\cdot 0}) = 0-(-1) = 1$$

$$P[X > 1, Y < 1] = ?$$

$$P[X < Y] = ?$$

What are the marg. p.s of X and Y ?