Revision: Random Variables $\mathcal{X}: S \to \mathbb{R}$

Probabilities $p(x_i)$ for each single value x_i that $\mathcal{X}$ can take

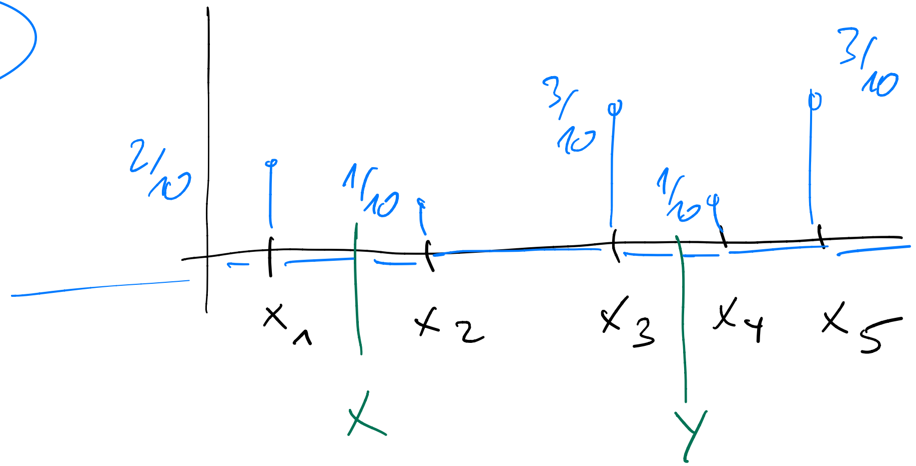
$$0 < p(x_i) < 1$$

$$\sum_i p(x_i) = 1$$

p is the probability mass function (pmf)

Distinguish 2 kinds

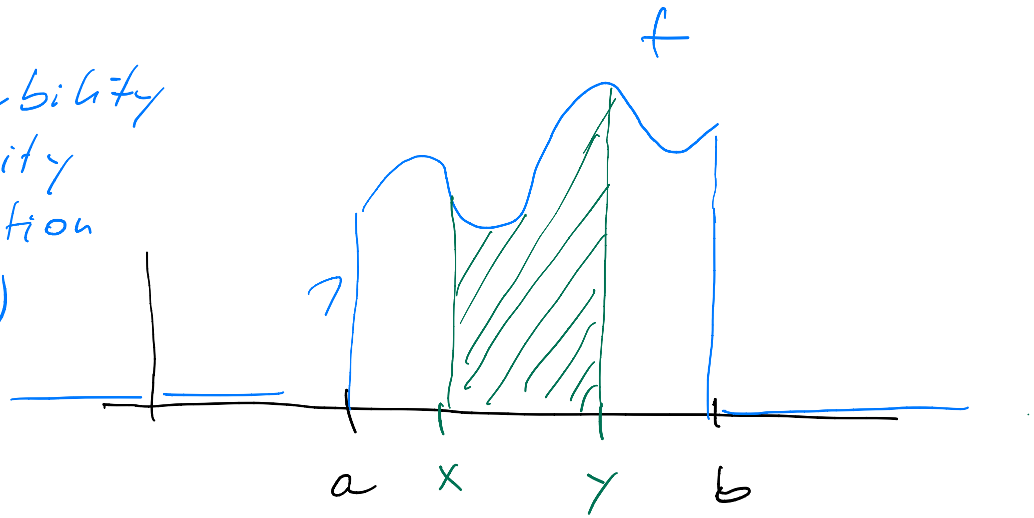
discrete



$$P[x < \mathcal{X} \le y] = \sum_{x < x_i \le y} p(x_i)$$

continuous

probability density function (pdf)

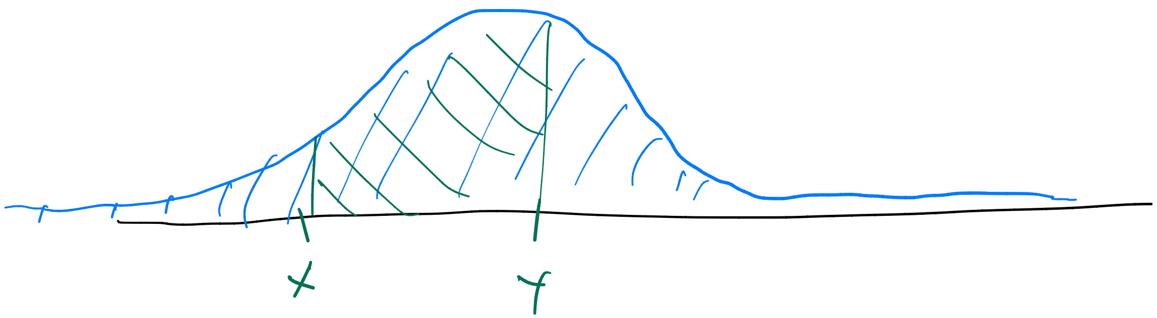


$$\int_a^b f(z)\,dz = 1 \quad , \quad f(x) \ge 0$$

$$\int_{-\infty}^{\infty} f(z)\,dz = 1$$

$$P[x < \mathcal{X} \le y] = P[x \le \mathcal{X} \le y] = \int_x^y f(z)\,dz$$

$$\int_{-\infty}^{\infty} f(z)\,dz = 1$$



In continuous case

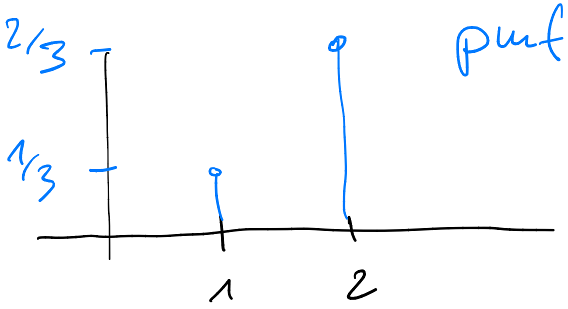
$$P[\mathcal{X} = x] = \int_x^x f(z)\,dz = 0$$

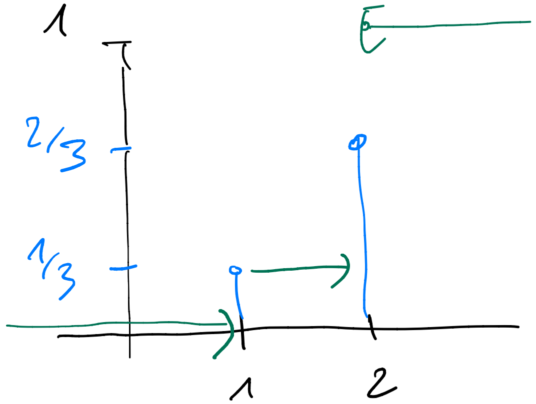
i.e., a single value has probability 0

Distribution Function

$$F(x) = P[X \le x]$$

discrete case





$$F(x) = \sum_{x_i \le x} p(x_i)$$

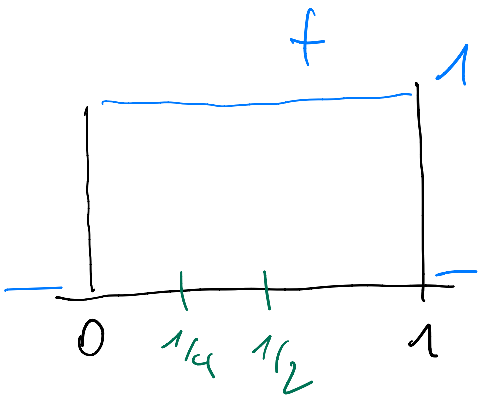
$$F(x) = \int_{-\infty}^{x} f(z)\,dz$$

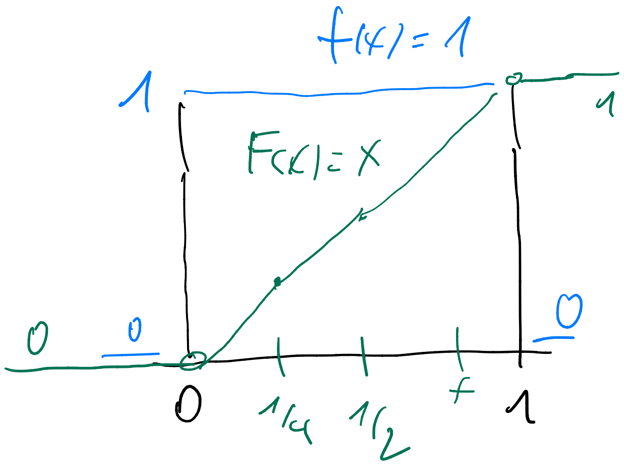
continuous case

Uniform distribution $U[0,1]$
(special case of $U[a,b]$)

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Always: $F' = f$
(from Fundamental Theorem of Calculus)





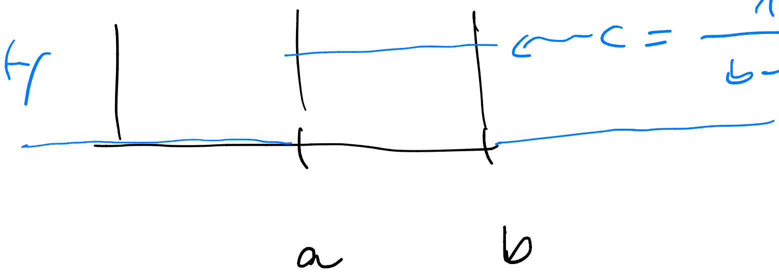
Here, $F(x) = \int_0^x f(z)\,dz$
$= \int_0^x 1\,dz = x$
since f is 0 on $[-\infty, 0]$

Examples of Continuous Distributions

1) Uniform Distribution: "Waiting for bus"

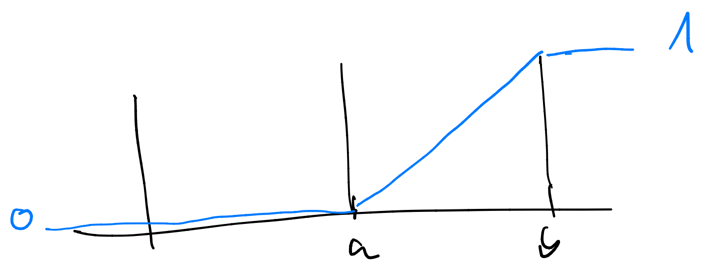
$U[a,b]$: uniform on $[a,b]$

density



$c = \frac{1}{b-a} = f(x), \; a \le x \le b$

distribution



2 Exponential Distribution

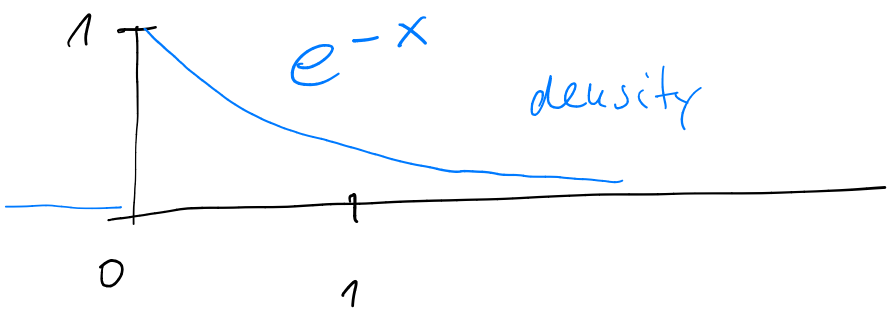
Waiting for: a customer, an email, an atom to decay

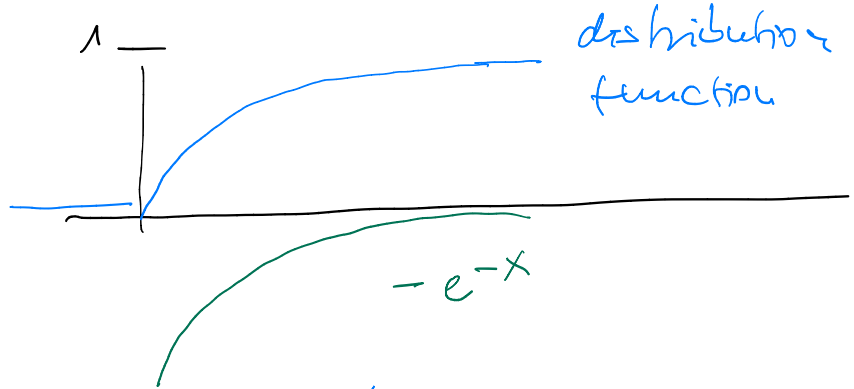
"Process w/o memory":

The probability to wait at least a minutes does not depend on having waited already b minutes

$$P[X > a] = P[X > a+b \mid X > b]$$

This leads to the exponential distribution,





$1 - e^{-x}, \quad x \geq 0;$

0 otherwise

Check: the derivative of the distribution is the density

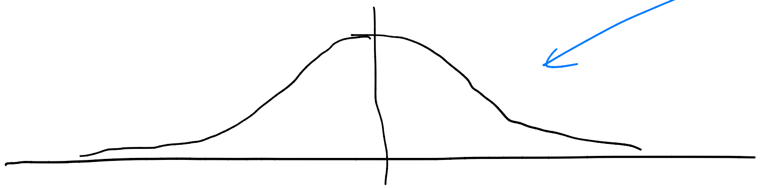
$$\frac{d}{dx} 1 - e^{-x} = 0 - e^{-x}(-1) = e^{-x}$$

3. Normal Distribution $\mathcal{N}(0, \frac{1}{2})$

mean variance

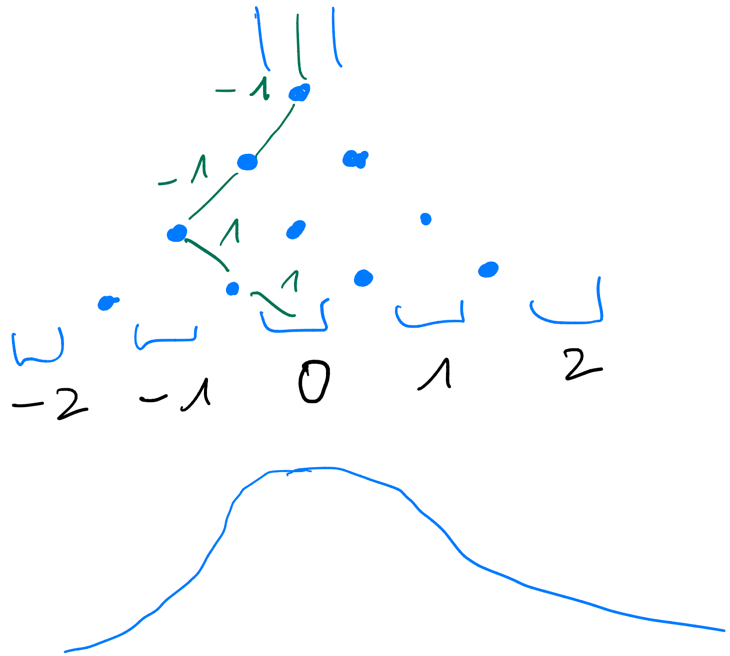
$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

density of $\mathcal{N}(0, \frac{1}{2})$

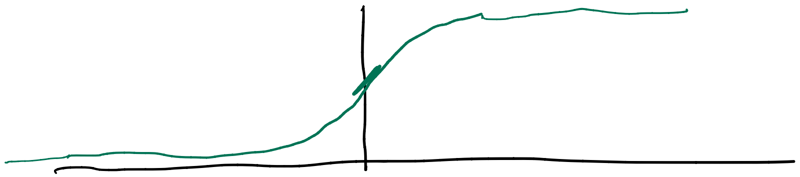


Sum of independent RVs, with same distribution,

e.g. Galton Board



$\mathcal{N}(0, \frac{1}{2})$ distribution



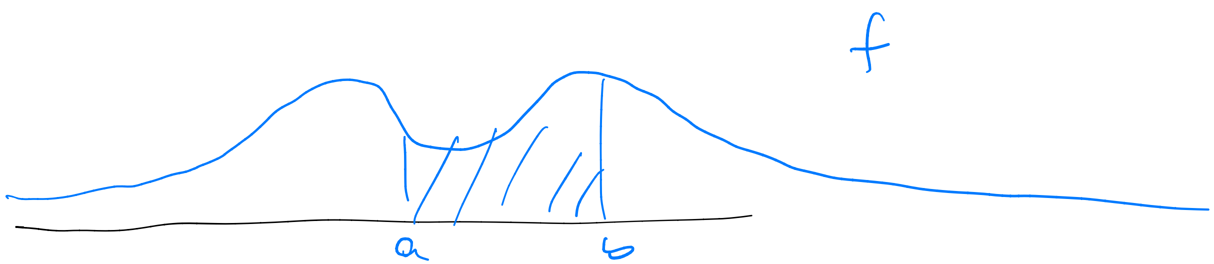
Distribution function of the normal distribution is written

Φ for $\mathcal{N}(0,1)$

Density vs. Distribution (Student Question)

$F(x) = P[X \le x]$ is the definition of the distribution function of X

Suppose f is the density of X (X continuous)



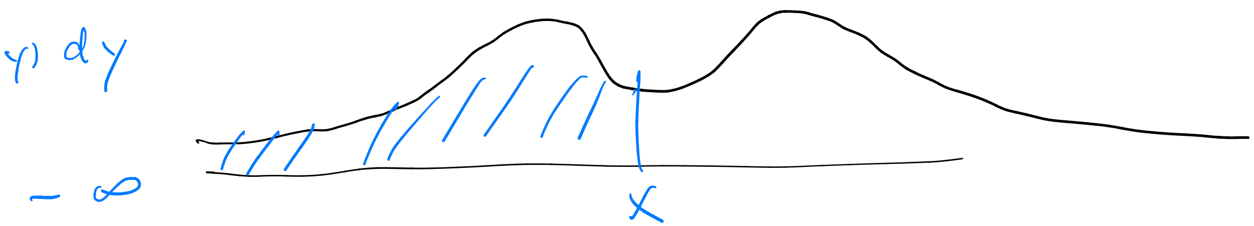
Then $f \ge 0$,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

"f is density of" means "$P[a \le X \le b] = \int_a^b f(x)\,dx$"

What is F in this case?

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(y)\,dy$$



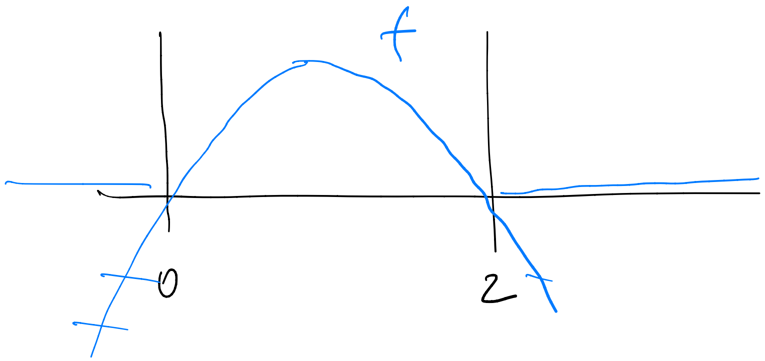
Exercice (Quiz)

$$f(x) = \begin{cases} c\,(2x - x^2), & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = c \quad x\,(2-x)$$

$$= -c \quad (x-0)(x-2)$$

Which c turns f into a density?



We want

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Here,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 f(x)\,dx = \int_0^2 c\,(2x - x^2)\,dx$$

$$= c \int_0^2 2x - x^2\,dx = c\left(\int_0^2 2x\,dx - \int_0^2 x^2\,dx\right)$$

$$= c\left(\left[x^2\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2\right) = c\left((2^2 - 0^2) - \left(\frac{2^3}{3} - \frac{0^3}{3}\right)\right)$$

$$= c\left(4 - \frac{8}{3}\right) = c\left(\frac{12}{3} - \frac{8}{3}\right) = c\,\frac{4}{3} = 1 \Rightarrow c = \frac{3}{4}$$

2.4 Expectation

Throwing a die: let X be the value of the die. On "average" we see

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

Tossing a coin n times: Expected # of heads:

$$1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2} \quad \text{if we toss once}$$

$n = 2$

$$0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

In general: $\quad \frac{n}{2} = n \cdot \frac{1}{2}$

Tossing the coin until first head:

$$\sum \frac{1}{n}$$

not converging..

$$1 \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{2}\right)^2 + 3 \cdot \left(\frac{1}{2}\right)^3 + \dots$$

head 1st time

T, H

T, T, H

$$= \sum_{k=1}^{\infty} k \cdot \left(\frac{1}{2}\right)^k =$$

Geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \qquad |x|<1$$