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3 Integrals

The questions below are about integrals that play an important role in probability theory. The first question asks for indefinite integrals and the later ones for definite integrals that allow one to determine crucial parameters of the uniform, the exponential and the normal distribution.
When working on the questions, remember the rules for integration, derived from the ones for derivatives. The two most important are the one of integration by parts and the substitution rule.

Integration by Parts: As a long version it reads

$$\int f'(x) \cdot g(x)\, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\, dx.$$

and as a short version:

$$\int f' \cdot g = f \cdot g - \int f \cdot g'.$$

Substitution: The rule can be applied in two ways, as a *forward version*

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(y)\, dy\, |_{y=g(x)}$$

and a *backward version,* which requires that the function g has an inverse:

$$\int f(x)\, dx = \int f(g(y)) \cdot g'(y)\, dy\, |_{y=g^{-1}(x)}.$$

Recall as well that there is an efficient technique to adapt the choice of f and g to match these patterns: Introduce y such that $y = g(x)$, determine dy/dx, and use this to substitute $g'(x)dx$ by dy, as shown in the lecture.

3.1 Indefinite Integrals

Determine the following indefinite integrals, that is, find antiderivatives for the functions under the integrals sign:

1. $\int 1\, dx$
2. $\int x\, dx$
3. $\int x^2\, dx$

4. $\int e^{-x}\,dx$
5. $\int e^{-\lambda x}\,dx$, where $\lambda > 0$
6. $\int x\,e^{-x}\,dx$
7. $\int x^2\,e^{-x}\,dx$
8. $\int x\,e^{-x^2}\,dx$
9. $\int x\,e^{-\lambda x}\,dx$, where $\lambda > 0$
10. $\int x^2\,e^{-\lambda x}\,dx$, where $\lambda > 0$.

Work first on problems 1.) to 8.). Go back to 9.) and 10.) only after you finished Question 3.2 and Parts 1 and 2 of Question 3.3.

3.2 Parameters of the Uniform Distribution

In this exercise you are asked to compute important parameters of the uniform distribution. Suppose that a, b are real numbers, $a < b$. The uniform distribution with parameters a and b, denoted as $U[a, b]$, associates to each subinterval of $[a, b]$ a probability that is proportional to its length. Its density is therefore a function $f\colon \mathbb{R} \to \mathbb{R}$ that is equal to a constant $c > 0$ on $[a, b]$ and 0 otherwise, that is,

$$f(x) = \begin{cases} c & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

The goal of the following questions is to determine the constant c, the mean value and second moment of $U[a, b]$.

1. Calculate
$$\int_a^b 1\,dx.$$
2. For which constant $c > 0$ is it that case that
$$\int_a^b c\ dx = 1\ ?$$

In the following, let c be the constant calculated in Part 2.

3. Calculate the mean of the uniform distribution $U[a, b]$:
$$\int_a^b c\,x\ dx.$$
4. Calculate the second moment of the uniform distribution $U[a, b]$:
$$\int_a^b c\,x^2\ dx.$$

3.3 Parameters of the Exponential Distribution

In this exercise you are asked to compute parameters analogous to the ones in Question 3.2 for the exponential distribution. Suppose that $\lambda > 0$ is a real number. The exponential distribution with parameter λ is denoted as *Exp*(λ) and its density function is $f\colon \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} c\,e^{-\lambda x} & 0 \leq x \\ 0 & \text{otherwise.} \end{cases}$$

The goal of the following questions is to determine the constant c, the mean value and second moment of *Exp*(λ).

1. Calculate

$$\int_0^\infty e^{-\lambda x}\,dx.$$

2. For which constant $c > 0$ is it that case that

$$\int_0^\infty c\,e^{-\lambda x}\,dx = 1?$$

In the following, let c be the constant calculated in Part 2.

3. Calculate the mean of the exponential distribution *Exp*(λ):

$$\int_a^b c\,x\,e^{-\lambda x}\,dx.$$

4. Calculate the second moment of the exponential distribution *Exp*(λ):

$$\int_a^b c\,x^2\,e^{-\lambda x}\,dx.$$

3.4 Computers Breaking Down

The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda\,e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1. What is the probability that a computer will function between 50 and 150 hours before breaking down?

   Hint: Note that first you have to determine λ so that f is a density.

2. What is the probability that it will function less than 100 hours?