# Derivatives and Integrals – Revision

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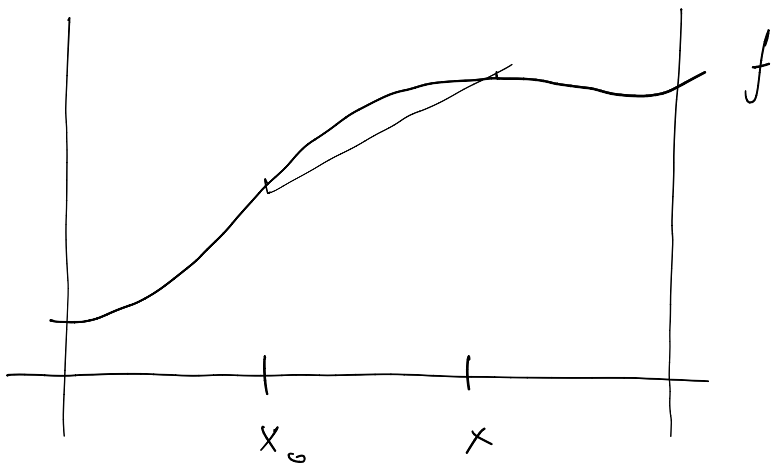
# Derivatives

Consider a function $f : [a,b] \longrightarrow \mathbb{R}$.

We want to define what is the steepness of the curve $f$

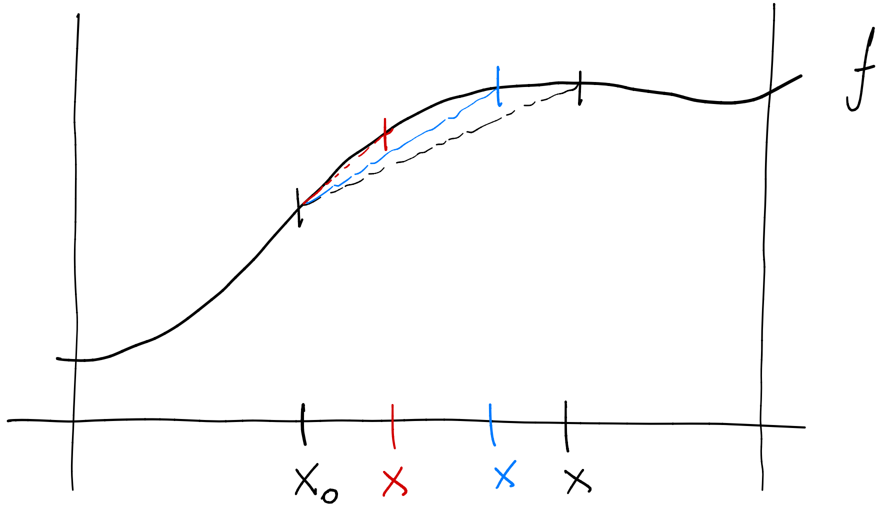



If $x_0 < x$, we can say what is the steepness over the stretch from $x_0$ to $x$: it is the gain in height divided by the length of the stretch:

$$\frac{f(x) - f(x_0)}{x - x_0}$$

What happens if we choose $x$ ever closer to $x_0$? The result depends more and more on the immediate environment of $x_0$. If the limit of the quantity

$$\frac{f(x)-f(x_0)}{x-x_0}$$




exists then we can see it as the steepness of $f$ in position $x_0$.

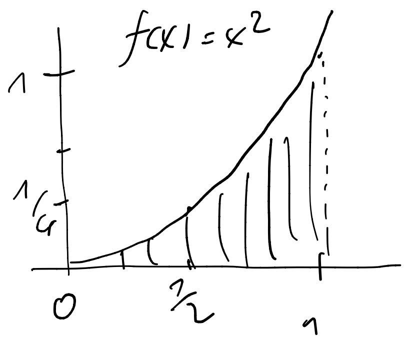
We call it the derivative of $f$ in $x_0$ and denote it as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

# Integrals: Area under the Curve

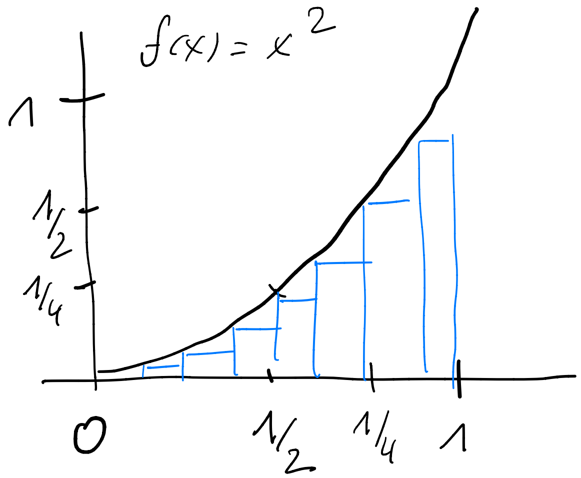
A question arising frequently in Geometry, Physics, and Engineering is to determine the area below the graph of a function.

Consider for example the function $f(x) = x^2$ and suppose we want to find out the area between the x-axis from 0 to 1 in our coordinate system and the graph of $f$:

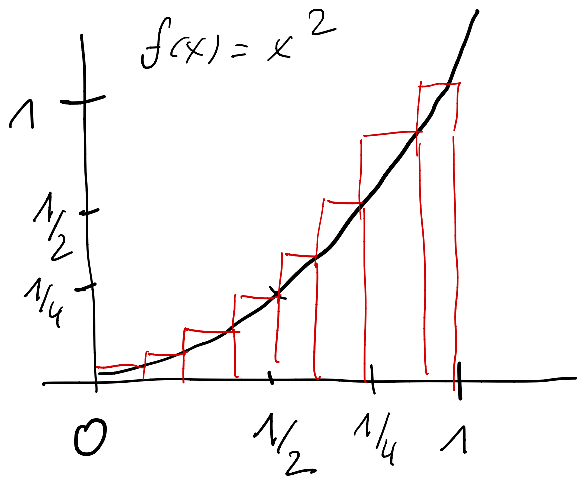



We would like to do that for as general a class of functions as possible.

# Idea 1: Brute Force

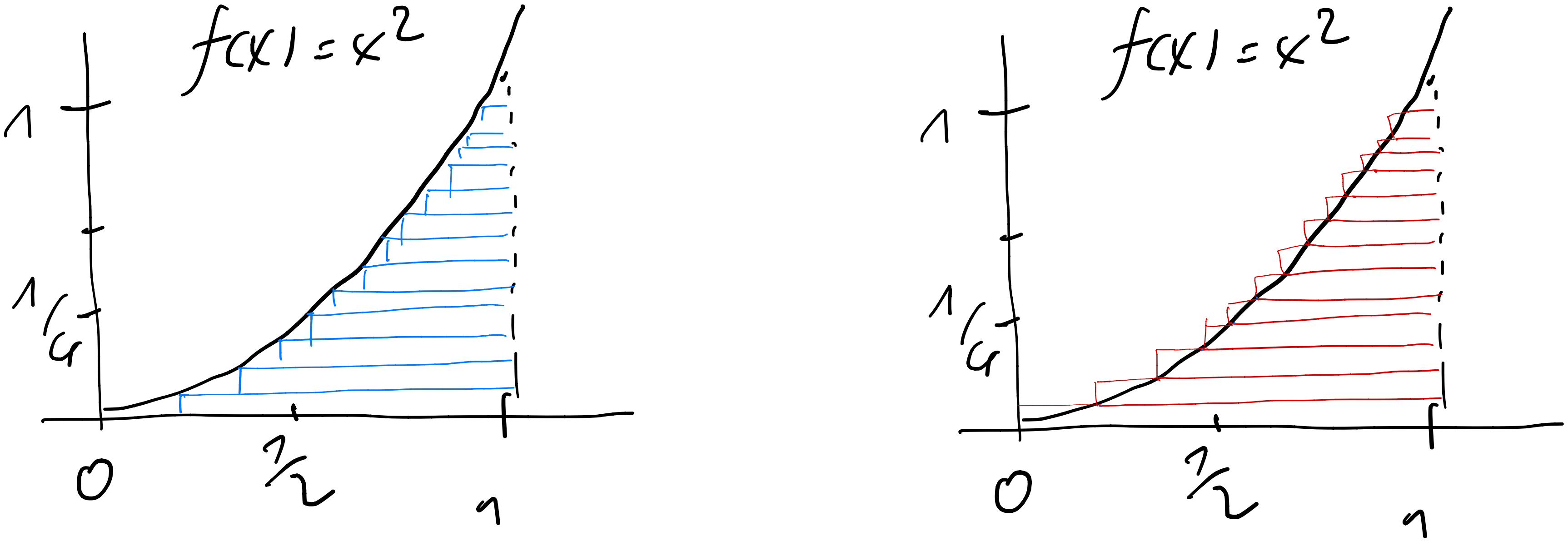



- Approximate the area from below and above by putting very narrow rectangles above and below the curve.
- The sum of the areas of rectangles is conceptually easy to compute.
- The narrower the rectangles, the greater the lower approximation and the smaller the upper approximation.



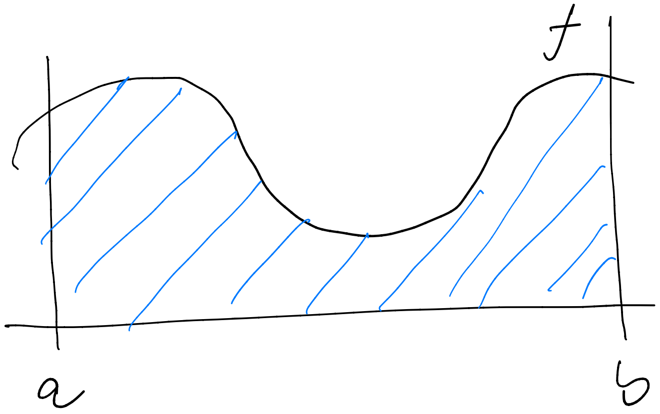

- If both approximations have the same limit, we say that is the area.
- The definition of the Riemann / Darboux integral makes this formal.

- There are other ways to define the area under the curve: Lebesgue approximated the area from above and below by horizontal bars:




- However, to get the main idea of integrals, the formal definition of areas is not important.
- Leibniz published the Fundamental Theorem of Differential and Integral Calculus in 1693, Riemann presented his definition in 1854.

We assume now that for "reasonable" functions $f$ we have an intuitive understanding of what is the area below $f$ from $a$ to $b$:




We call this area the integral of $f$ from $a$ to $b$ and write it

This is like a big S for "sum"!

$$\int_a^b f(x)\,dx \quad , \quad \text{e.g.,} \quad \int_0^1 x^2\,dx \, .$$

In an expression like $\int_1^2 ax^2y^3\,dx$, the $x$ in $dx$ tells us what is the variable of the function.

How are

$$f \quad \text{and} \quad \int_a^b f(x)\,dx$$

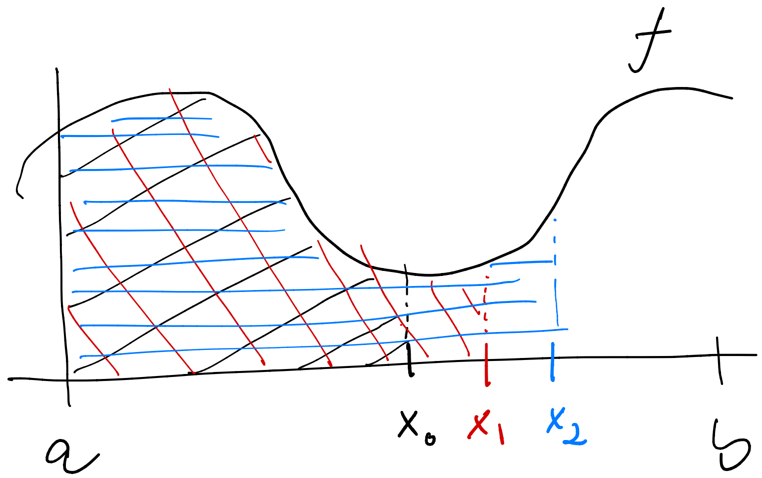
related?

**Idea 2: Generalize!**

We generalize the problem of finding the area under $f$ from $a$ to $b$ to the one of finding the area from $a$ to arbitrary $x$'s:

$$F(x) := \int_a^x f(x)\,dx$$

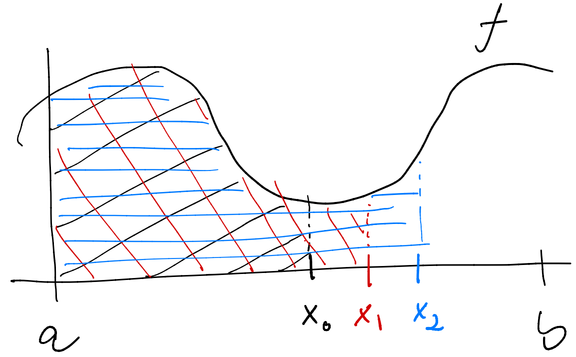
The figure shows $F(x_0)$, $F(x_1)$, $F(x_2)$



If we know $F$, then we also know the area under $f$ from some $x_0 \in [a,b]$ to some $x_1 \in [x_0, b]$: It is

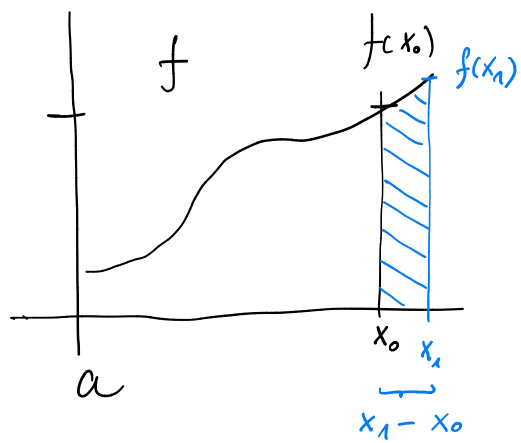
$$\int_{x_0}^{x_1} f(y)\,dy = \int_a^{x_1} f(y)\,dy - \int_a^{x_0} f(y)\,dy$$

$$= F(x_1) - F(x_0)$$




We study how steep is $F$: from $x_0$ to $x_1$ the slope of $f$ is

$$\frac{F(x_1) - F(x_0)}{x_1 - x_0}.$$

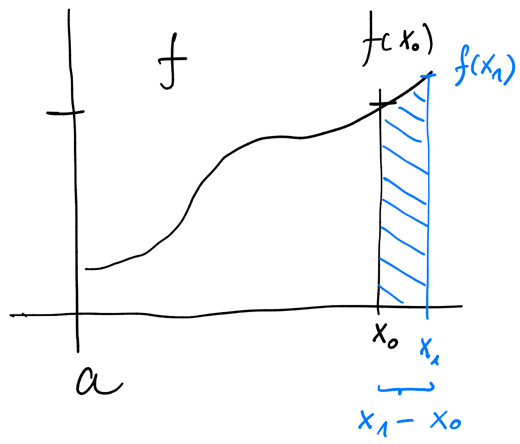



This has a geometric interpretation:

It is the area of the blue column divided by the width $(x_1 - x_0)$.

Suppose that $f$ does not vary too much (i.e., $f$ is continuous).

Then $f(x_1)$ gets ever closer to $f(x_0)$ as $x_1$ approaches $x_0$ and the blue area, which has size $F(x_1) - F(x_0)$, gets ever closer to the rectangle with height $f(x_0)$ and width $x_1 - x_0$, that is,




$$F(x_1) - F(x_0) \approx f(x_0)(x_1 - x_0).$$

and in the limit we have

$$\lim_{x_1 \to x_0} \frac{F(x_1) - F(x_0)}{x_1 - x_0} = f(x_0),$$

in short

$$F' = f.$$

(This is the essence of Leibniz' Fundamental Theorem of Calculus.)

Question: How well do we know a function if we know its derivative?

Let $G$ be another function with $G' = f$. Then

$$(G-F)' = G' - F' = f - f = 0.$$

We already know, if $g(x) = \text{const}$, then $g' = 0$.

Also the converse holds:
If $g'(x) = 0$ for all $x$ in $[a,b]$, then there is a constant $c \in \mathbb{R}$ s.th. $g(x) = c$ for all $x \in [a,b]$.
(proved by Mean Value Theorem)

We conclude: there is a constant $c \in \mathbb{R}$ s.th.

$$G = F + c.$$

Suppose $G' = f$. How can we calculate integrals with $G$?

We had $F(x) = \int_a^x f(y)\,dy$

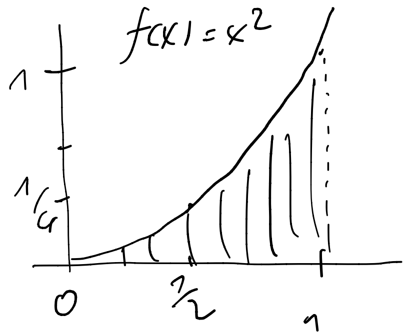
$$\Rightarrow F(a) = \int_a^a f(x)\,dx = 0$$ area of width 0

Now,

$$\underline{\int_a^x f(y)\,dy} = F(x) = F(x) - F(a)$$

"17 camels trick" $\rightsquigarrow$ $= F(x) + c - (F(a) + c) = \underline{G(x) - G(a)}$

Application: $\int_0^1 x^2 \, dx$




Here: $f(x) = x^2$

If $G(x) = \frac{x^3}{3}$, then

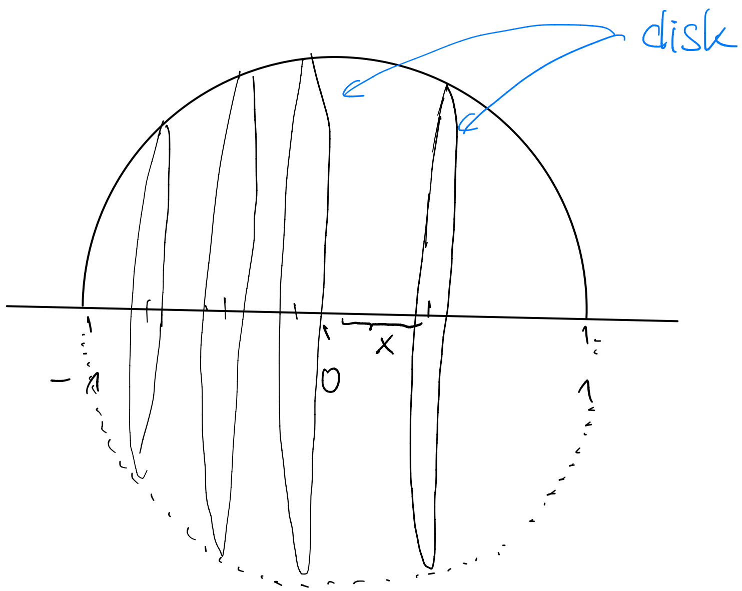
$$G'(x) = \frac{d}{dx}\frac{x^3}{3} = \frac{1}{3}\frac{d}{dx}x^3 = \frac{1}{3}\,3x^2 = x^2 = f(x)$$

Therefore

$$\int_0^1 x^2 \, dx = G(1) - G(0) = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

# Application: Volume of a Sphere of Radius 1

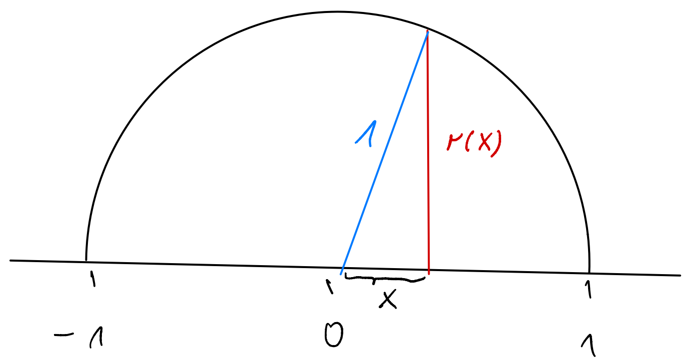
- We can think of the sphere as consisting of infinitely many disks stacked one to the next.




- The disk at position $x$, $x \in [-1,1]$, has area $f(x) = \pi\, r(x)^2$.
- The volume of the sphere is then

$$V = \int_{-1}^{1} \pi\, r(x)^2 \, dx.$$

- By Pythagoras, $x^2 + r(x)^2 = 1$.
  $\Rightarrow r(x)^2 = 1 - x^2$.




- If $G(x) = \pi\left(x - \frac{x^3}{3}\right)$, then $G'(x) = f(x)$.

$$\Longrightarrow \underline{V} = \int_{-1}^{1} \pi(1-x^2)\,dx = G(1) - G(-1) = \pi\left(1 - \frac{1^3}{3} - \left(-1 - \frac{(-1)^3}{3}\right)\right)$$

$$= \pi\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \pi\,\frac{4}{3} = \underline{\frac{4}{3}\pi}$$

## Consequences

- If we want to know the area under $f$ from $a$ to $b$, we can find a function $F$ such that $F' = f$. Then the area is
$$\int_a^b f(x)\,dx = F(a) - F(b)$$

- If we want to find a function $G$ such that
  - $G' = f$
  - $G(a) = c$

and find $G(b)$ for some $b$, then we know $G(b) = c + \int_a^b f(x)\,dx$.

If we have a way to compute the area under $f$ from $a$ to $b$, then we can compute $G(b)$. Often, the area can be approximated.

# How can we Calculate Integrals?

Need to solve the problem

Input: $f$

Output: $F$ sth $F' = f$

Such an $F$ is called an <u>antiderivative</u> of $f$

Investigate:

- How to compute derivatives $f'$ from $f$

- How to "reengineer" $f$ from $f'$

Approach:

- Study laws of derivatives to find laws of integration.

## Laws of Derivation 1: Multiplication by a constant

Suppose $f$ has a derivative. Consider $g(x) = c \cdot f(x)$, $c \in \mathbb{R}$
Then

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c \cdot f(x) - c \cdot f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} c \cdot \frac{(f(x) - f(x_0))}{x - x_0} = c \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = c \cdot f'(x_0)$$

Here we used that constants can be pulled out of a limit.
The calculation gives us the property

$$\boxed{(c \cdot f)' = c \cdot f',}$$

that is, multiplicative constants can be pulled out of the derivative.

# Continuity of Addition and Multiplication

Recall that limits are generally compatible with addition and multiplication:

$$\lim_{x \to x_0} \big( f(x) + g(x) \big) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

$$\lim_{x \to x_0} \big( f(x) \cdot g(x) \big) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

These properties are also known as "continuity of addition and multiplication.

## Laws of Derivation 2: Derivative of a Sum

We next apply the continuity of addition to derivatives.

Suppose $h(x) = f(x) + g(x)$. Then

$$h'(x_0) = \lim_{x \to x_0} \frac{f(x) + g(x) - (f(x_0) + g(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(f(x) - f(x_0)) + (g(x) - g(x_0))}{x - x_0}$$

continuity of +

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) + g'(x_0)$$

Hence $\boxed{(f+g)' = f' + g'}$

# Laws of Derivation 3: Product Rule

To derive a rule for products, we will use the trick of adding and subtracting a useful term (generally known as "17 camels trick").

Suppose $h(x) = f(x) \cdot g(x)$. Then

$$h'(x_0) = \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} g(x) + f(x_0) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

Hence, $\boxed{(f \cdot g)' = f'g + fg'.}$

In Leibniz notation, this is

$$\frac{d}{dx} f(x)\,g(x) = \left(\frac{d}{dx} f(x)\right) g(x) + f(x)\left(\frac{d}{dx} g(x)\right)$$

Application: Derivatives of polynomials

Applying the "product rule" to $x^2$ yields

$$\frac{d}{dx} x^2 = \frac{d}{dx} x \cdot x = \left(\frac{d}{dx} x\right) \cdot x + x \cdot \left(\frac{d}{dx} x\right)$$

$$= 1 \cdot x + x \cdot 1 = 2x$$

We can also show by induction that $\frac{d}{dx} x^n = n x^{n-1}$:

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx} x \cdot x^n = \left(\frac{d}{dx} x\right) \cdot x^n + x \cdot \left(\frac{d}{dx} x^n\right)$$

$$= 1 \cdot x^n + x \cdot n x^{n-1} = x^n + n x^n = (n+1) x^n$$

We conclude: $\frac{d}{dx} \frac{x^{n+1}}{n+1} = \frac{(n+1) x^n}{n+1} = x^n$

" $\frac{x^{n+1}}{n+1}$ is an antiderivative of $x^n$ "

# Laws of Derivation 4: Chain Rule

Chain Rule: We now consider a function that is the composition of two functions. Suppose $h(x) = f(g(x))$.

Then

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(g(x_0)) - f(g(x))}{x - x_0}$$

"17 camels trick"

$$= \lim_{x \to x_0} \frac{f(g(x_0)) - f(g(x))}{g(x_0) - g(x)} \cdot \frac{g(x_0) - g(x)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(g(x_0)) - f(g(x))}{g(x_0) - g(x)} \cdot \lim_{x \to x_0} \frac{g(x_0) - g(x)}{x - x_0}$$

If $g$ is differentiable, then $g$ is continuous; then $g(x) \to g(x_0)$ if $x \to x_0$.

$$= \lim_{y \to g(x_0)} \frac{f(g(x_0)) - f(y)}{g(x_0) - y} \cdot g'(x_0)$$

$$= f'(g(x_0)) \cdot g'(x_0)$$

$$\boxed{(f \circ g)' = (f' \circ g) \cdot g'}$$

## Application : $\frac{d}{dx} \sin(x^2)$

Let us suppose we know that $\sin' = \cos$, that is, cosine is the derivative of sine.

What is the derivative of $\sin(x^2)$ ?

We can match the chain rule viewing $f$ as $f(y) = \sin(y)$ and $g(x) = x^2$. Then $f'(y) = \cos(y)$ and $g'(x) = 2x$.

Then the chain rule tells us

$$\frac{d}{dx} \sin(x^2) = \sin'(x^2) \cdot 2x = \cos(x) \cdot 2x = 2x \cos(x)$$

We can remember the rule as telling us to first take the derivative of the outer function and then multiplying it with the one of the inner function: "outer derivative times inner derivative"

# Laws of Derivation 5: Inverse Rule

Let $g$ be the inverse function of $f$,

that is,

$$f(g(x))=x \quad \text{and} \quad g(f(y))=y$$

For example, let $f:\mathbb{R}\longrightarrow\mathbb{R}^+$ be the exponential function $f(x)=\exp(x)$. Then $f$ has an inverse function, the (natural) logarithm $g:\mathbb{R}^+\longrightarrow\mathbb{R}$, $g(y)=\log(y)$.

Then

$$1=\frac{d}{dx}x=f(g(x))=f'(g(x))\cdot g'(x).$$

by chain rule

This gives us $\boxed{g'(x)=\frac{1}{f'(g(x))}}$

Note that not every function has an inverse. Note also that the existence of an inverse depends on what we assume to be domain and range of $f$.

## Application :

$$\frac{d}{dx} \log(x)$$

$f(x) = \exp(x)$

$g(x) = \log x$

Let's assume we know that $\exp'(x) = \exp(x)$.
Actually, one way to define exp is to require that

(i) $\exp' = \exp$  (ii) $\exp(0) = 1$

Then there is exactly one function that satisfies these requirements.

What is $\log'$ ?

By the inverse rule

since $\exp' = \exp$

$$\boxed{\log'(x)} = \frac{1}{\exp'(\log x)} = \frac{1}{\exp(\log x)} = \boxed{\frac{1}{x}}$$

since exp and log are inverses

# Laws of Derivation 6: Simple Quotient Rule

Suppose that $g(x) = \dfrac{1}{f(x)}$.

Consider $h(x) = \dfrac{f(x)}{f(x)} = f(x) \cdot g(x)$.

Then $h(x) = 1$. Hence $h'(x) = 0$. Therefore,

$$0 = h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\Rightarrow \quad f(x)\, g'(x) = -f'(x)\, g(x)$$

$$\Rightarrow \quad g'(x) = -\frac{f'(x)}{f(x)}\, g(x) = -\frac{f'(x)}{f(x)} \cdot \frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}$$

Hence,

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

# Integration

We distinguish different problems:

1. Finding an antiderivative, i.e. given $f$, find $F$ sth $F' = f$.

To express that $F$ is an antiderivative of $f$, we write

$$F(x) = \int f(x)\, dx$$

This is not very precise, since $f$ has infinitely many antiderivatives, which all differ by a constant. People often write therefore

$$\int f(x)\, dx = F(x) + c,$$

for instance $\int x^2 dx = \frac{x^3}{3} + c$

# Integration

We distinguish different problems:

1. Find an antiderivative or indefinite integral, i.e., given $f$

   find $F$ sth $F' = f$.

   To express that $F$ is an antiderivative of $f$, we write

$$F(x) = \int f(x)\,dx$$

   This is not very precise, since $f$ has infinitely many antiderivatives, which all differ by a constant. People often write therefore

$$\int f(x)\,dx = F(x) + c,$$

   for instance
$$\int x^2\,dx = \frac{x^3}{3} + c$$

2. Determine a definite integral, i.e., given $f, a, b$, find the numbers

$$\int_a^b f(x)\,dx .$$

for instance

$$\int_{-1}^{1} \pi(1-x^2)\,dx = \frac{4}{3}\pi$$

3. Determine an improper integral, e.g., given $f, a$, find the number

$$\int_a^\infty f(x)\,dx := \lim_{z\to\infty} \int_a^z f(x)\,dx$$

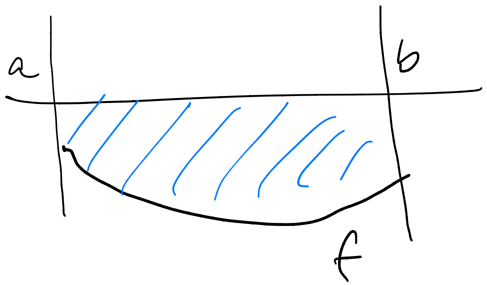
for instance

$$\int_1^\infty \frac{1}{x^2}\,dx$$

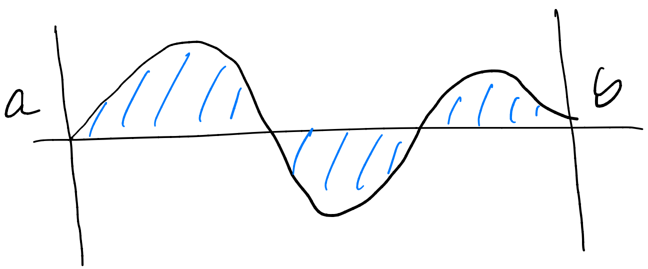
There are also $\int_{-\infty}^{b} f(x)\,dx$

and $\int_{-\infty}^{\infty} f(x)\,dx$

# What Does the Following Mean?




The integral of a negative function? What kind of area is that?




The integral of a function that is partly positive, partly negative?

$$\int_{1}^{-1} \pi(1-x^2)\,dx$$

An integral where we run backwards through our interval?

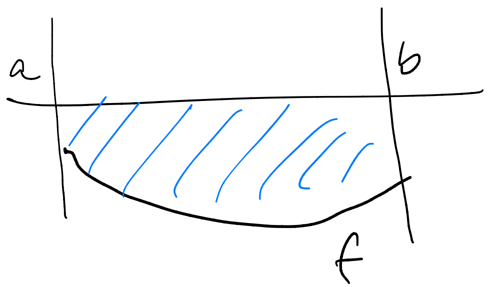
Intuition: Speedometer

- You sit in a moving car (on rails), you can't look out of the window, but you have a speedometer, and you know whether the car moves forward or backward.

- Can you find out at any moment where you are?

$$s'(t) = v(t)$$

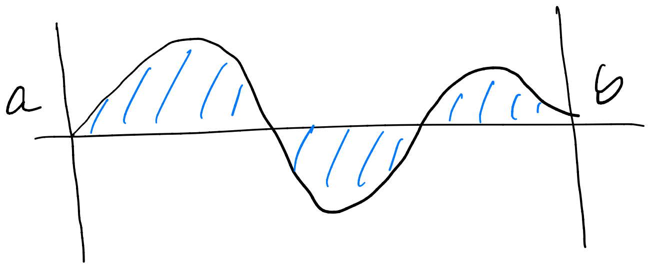
# What Does the Following Mean?




A car with negative velocity moves backward.

$\Rightarrow$ The integral is negative

$\Rightarrow$ Areas have a sign: they can be positive or negative




A car whose velocity changes between positive and negative moves forward and backward.

$\Rightarrow$ Positive areas are added up, negative areas are subtracted

$$\int_{1}^{-1} \pi(1-x^2)\,dx$$

We would like the following rule to hold:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

To make this work, we need that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

# Summary of Derivation Rules

$$(c\cdot f)' = c\cdot f'$$

$$(f+g)' = f' + g'$$

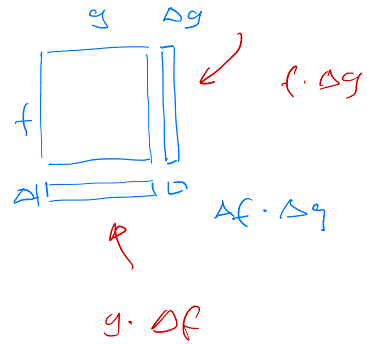
$$(f\cdot g)' = f'\cdot g + f\cdot g'$$

$$(f\circ g)' = (f'\circ g)\cdot g'$$

$$(f^{-1})' = \frac{1}{f'\circ f^{-1}}$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

Intuition for the product rule




Many functions that we see are composed of elementary functions:

$e^{x^2}$, $e^{-x^2}$,

$\sin(3t)$, $\sin(t^2)$

The chain rule says how to compute derivatives for them.

$$\frac{d}{dx}\sin(t^2)$$

$$= \cos(t^2)\cdot 2t$$

## Integration Rules: Suppose $f = F'$, $g = G'$

We derive them as mirrored versions of derivation rules

Remember: $F' = f \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$

$c \cdot f = c \cdot F' = (c \cdot F)'$ $\qquad$ $\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$

$f + g = F' + G' = (F + G)'$ $\qquad$ $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

# Integration by Parts Rule

We introduce a shorthand: For a function $h$ we write

$$\left[h(x)\right]_a^b := h(b) - h(a) = \int_a^b h'(x)\,dx$$

$$h(x)\Big|_a^b$$

$$(fg)' = f'g + fg'$$

$$\Rightarrow \quad f'g = (fg)' - fg'$$

$$\boxed{\int_a^b f'(x) \cdot g(x)\,dx}$$

$$= \int_a^b (f \cdot g)'(x)\,dx - \int_a^b f(x)\,g'(x)\,dx$$

$$= \boxed{\left[f(x) \cdot g(x)\right]_a^b - \int_a^b f(x)\,g'(x)\,dx}$$

$$\int f'g = \int (fg)' - \int fg'$$

$$\int f'g = fg - \int fg'$$

# Integration by Parts: Summary

$$\int_a^b f'(x) \cdot g(x)\, dx = \Big[ f(x) \cdot g(x) \Big]_a^b - \int_a^b f(x) \cdot g'(x)\, dx$$

# Example

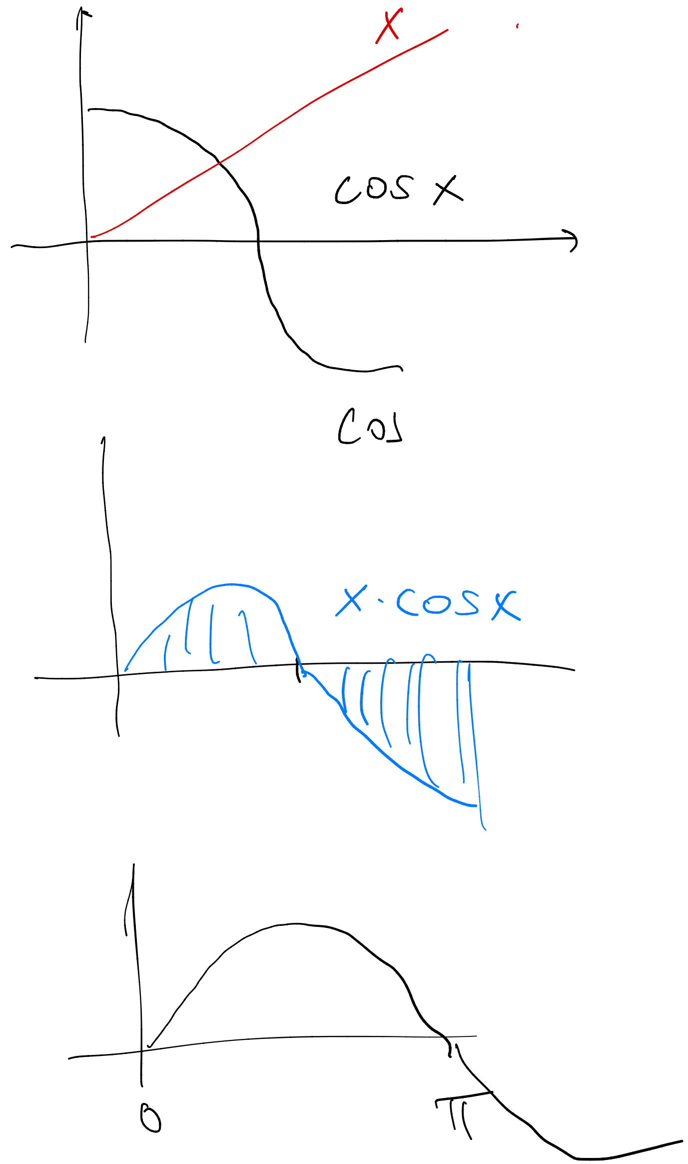
$$\int_0^\pi \overset{g}{x} \cdot \overset{f'}{\cos x} \, dx$$

$$= \Big[ x \cdot \sin x \Big]_0^\pi - \int_0^\pi \overset{g'}{1} \cdot \overset{f}{\sin x} \, dx$$

$$= (\pi \cdot \sin \pi - 0 \cdot \sin 0) - \Big[ -\cos x \Big]_0^\pi$$

$$= \pi \cdot 0 - 0 \cdot 0 + (\cos \pi - \cos 0)$$

$$= 0 + ((-1) - 1) = 0 + 0 = 0$$



# Integration by Parts: Example

$$\int_a^b f'(x)\cdot g(x)\,dx = \Big[f(x)\cdot g(x)\Big]_a^b - \int_a^b f(x)\cdot g'(x)\,dx$$

$$\int_0^\pi \overset{g}{x}\cdot\overset{f'}{\cos x}\,dx = \Big[\overset{g}{x}\cdot\overset{f}{\sin x}\Big]_0^\pi - \int_0^\pi \overset{g'}{1}\cdot\overset{f}{\sin(x)}\,dx$$

$$= (\pi\cdot\sin\pi - 0\cdot\sin 0) - \Big[-\cos x\Big]_0^\pi$$

$$= 0 + \Big[\cos x\Big]_0^\pi$$

$$= -1 - 1 = -2$$

## Integration by Substitution: Eliminate the Inner Function

$$(f \circ g) \cdot g' = (F \circ g)'$$

$$\int_a^b (f \circ g) \cdot g' = \int_a^b (F \circ g)' = \left[ F \circ g \right]_a^b = F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} f$$

$$\boxed{\int_a^b f(g(x)) \cdot g'(x)\, dx} = \left[ F(g(x)) \right]_a^b = \left[ F(y) \right]_{g(a)}^{g(b)}$$

$$= \boxed{\int_{g(a)}^{g(b)} f(y)\, dy}$$

## Example :

$$\int_0^{\sqrt{\pi}} x \cdot \cos(x^2)\, dx$$

$(f \circ g) \cdot g'$

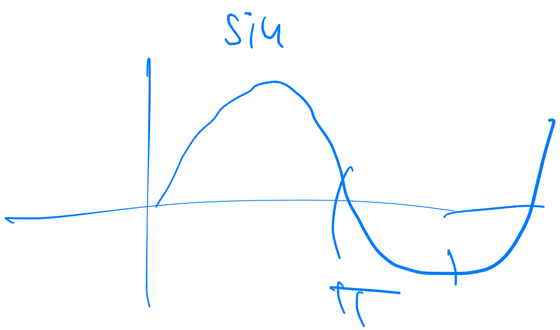
$$\int_0^{\sqrt{\pi}} x \cdot \cos(x^2)\, dx = \frac{1}{2} \int_0^{\sqrt{\pi}} \cos(x^2) \cdot 2x \, dx$$

$\frac{1}{2} 2x$

$g(x) = x^2$

$$= \frac{1}{2} \int_0^{\pi} \cos y \, dy$$

$$= \frac{1}{2} \left[ \sin y \right]_0^{\pi} = \frac{1}{2} (0 - 0) = 0$$



Example :
$$\int_0^{\sqrt{\pi}} x \cdot \cos(x^2)\, dx$$

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \Big[ F(g(x)) \Big]_a^b$$

$$\int_0^{\sqrt{\pi}} x \cdot \cos(x^2)\, dx = \frac{1}{2} \int_0^{\sqrt{\pi}} \overset{g'}{2x} \cdot \overset{f}{\cos}(\overset{g}{x^2})\, dx \qquad F(y) = \sin y$$

$$= \frac{1}{2} \Big[ \sin x^2 \Big]_0^{\sqrt{\pi}} = \frac{1}{2} \left( \sin(\sqrt{\pi}^2) - \sin 0 \right)$$

$$= \frac{1}{2} \left( \sin \pi - \sin 0 \right) = \frac{1}{2} (0 - 0) = 0$$

# Integration by Substitution: Eliminate the Inverse

We know

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \left[F(g(x))\right]_a^b = \left[F(y)\right]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(y)\,dy,$$

that is

$$\int_{g(a)}^{g(b)} h(y)\,dy = \int_a^b h(g(x))\cdot g'(x)\,dx$$

$$\int_1^4 \cos\sqrt{x}\,dx$$

$g^{-1}(x)$ where $g(y)=y^2$

$(f\circ g)\,g'$

Assume $h(y) = f(g^{-1}(y))$. Then

$$\boxed{\int_{g(a)}^{g(b)} f(g^{-1}(y))\,dy} = \int_a^b f(g^{-1}(g(x)))\cdot g'(x)\,dx$$

$$= \boxed{\int_a^b f(x)\cdot g'(x)\,dx}$$

Example: $\int_1^4 \cos\sqrt{y}\, dy$

Rule: $\int_{g(a)}^{g(b)} f(g^{-1}(y))\, dy = \int_a^b f(x)\cdot g'(x)\, dx$

$$\int_1^4 \cos\sqrt{y}\, dy = \int_{\sqrt{1}}^{\sqrt{4}} \cos(x)\cdot 2x\, dx$$

$g(x) = x^2$,

$g'(x) = 2x$

$g^{-1}(y) = \sqrt{y}$

$$= \Big[2x\cdot \sin x\Big]_1^2 - \int_1^2 2\cdot \sin x\, dx$$

$$= 2\cdot 2\cdot \sin 2 - 2\cdot 1\cdot \sin 1 + \Big[2\cos x\Big]_1^2$$

$$= \quad + 2(\cos 2 - \cos 1)$$

# Integration by Substitution: Example

Rule: $$\int_{g(a)}^{g(b)} f(g^{-1}(y))\,dy = \int_a^b f(x)\cdot g'(x)\,dx$$

or $$\int_a^b f(g^{-1}(x))\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(y)\cdot g'(y)\,dy$$

What is $\displaystyle\int_1^4 \underbrace{\cos}_{f}(\underbrace{\sqrt{x}}_{g^{-1}})\,dx$ ?

$g^{-1}(x) = \sqrt{x}$, $g(y) = y^2$

$g'(y) = 2y$

$$\int_1^4 \cos(\sqrt{x})\,dx = \int_{\sqrt{1}}^{\sqrt{4}} \cos(y)\cdot 2y\,dy$$

# Integration by Substitution: Example (cntd.)

$$\int_1^4 \cos(\sqrt{x})\,dx = \int_{\sqrt{1}}^{\sqrt{4}} \overset{f}{2y} \cdot \overset{g'}{\cos(y)}\,dy$$

apply integration by parts

$$= \Big[\overset{f}{2y} \cdot \overset{g}{\sin(y)}\Big]_1^2 - \int_1^2 \overset{f'}{2} \cdot \overset{g}{\sin(y)}\,dy$$

$$= \big(2 \cdot 2 \cdot \sin(2) - 2 \cdot 1 \cdot \sin(1)\big) - 2\Big[-\cos(y)\Big]_1^2$$

$$= 4 \cdot \sin(2) - 2 \cdot \sin(1) + 2\cos(2) - 2 \cdot \cos(1)$$

# Inverse Functions

Examples of inverse functions:

$$\exp^{-1} = \log$$

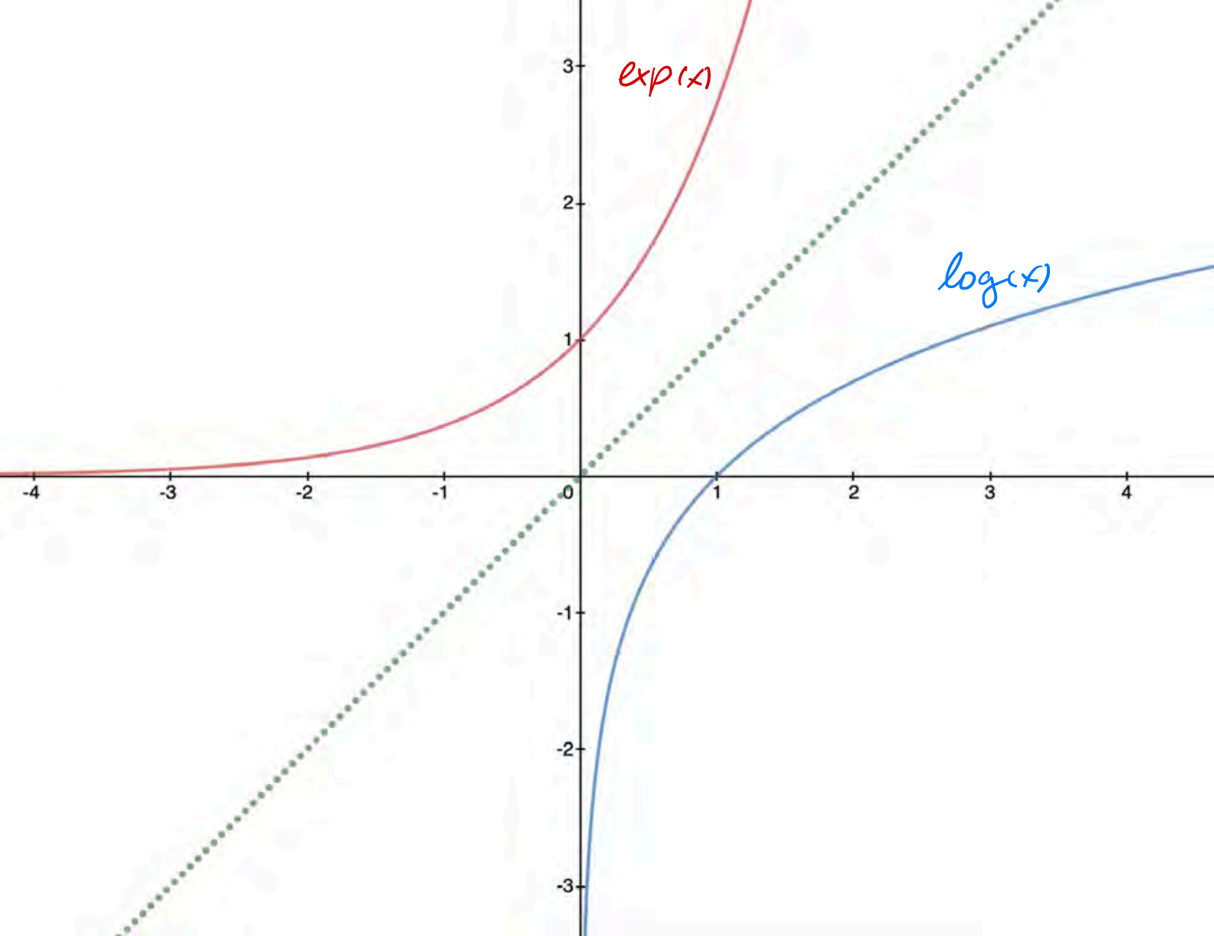
$$\mathrm{sqr}^{-1} = \mathrm{sqrt} \quad \text{where } \mathrm{sqr}(x) = x^2,\ \mathrm{sqrt}(y) = \sqrt{y}$$

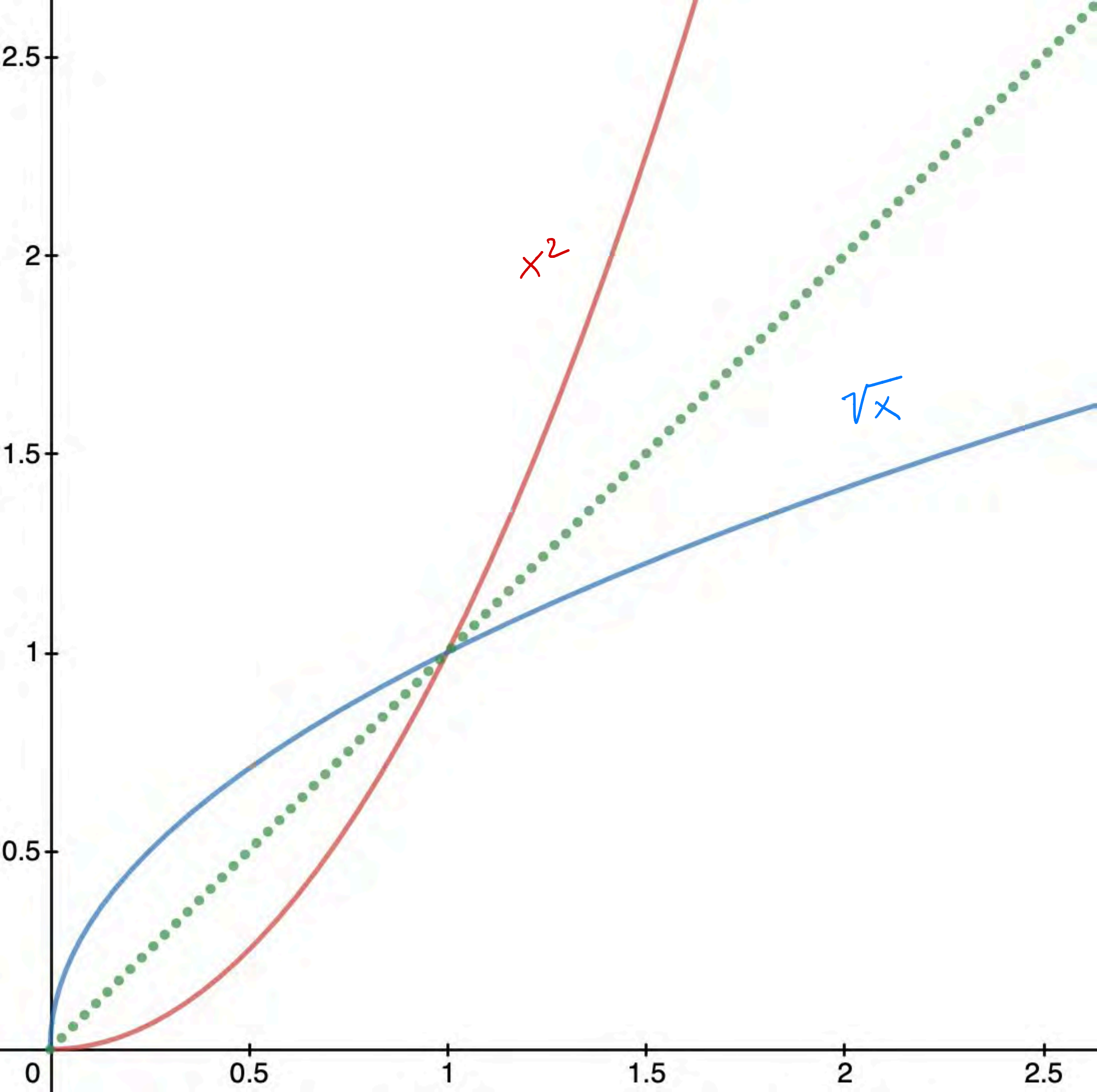
$$\sin^{-1} = \arcsin$$

$$\cos^{-1} = \arccos$$

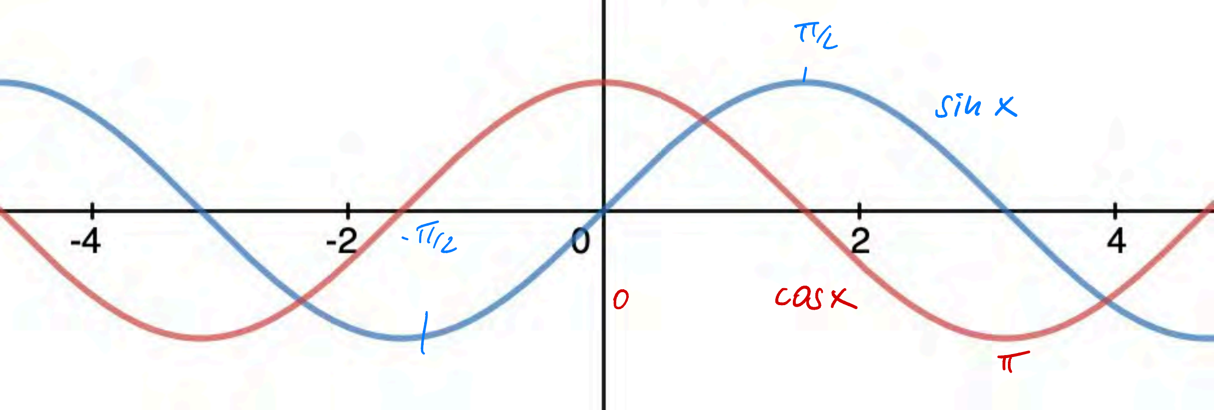
$$\tan^{-1} = \arctan$$

Also $\sqrt[3]{y}$ is the inverse of $x^3$, ...

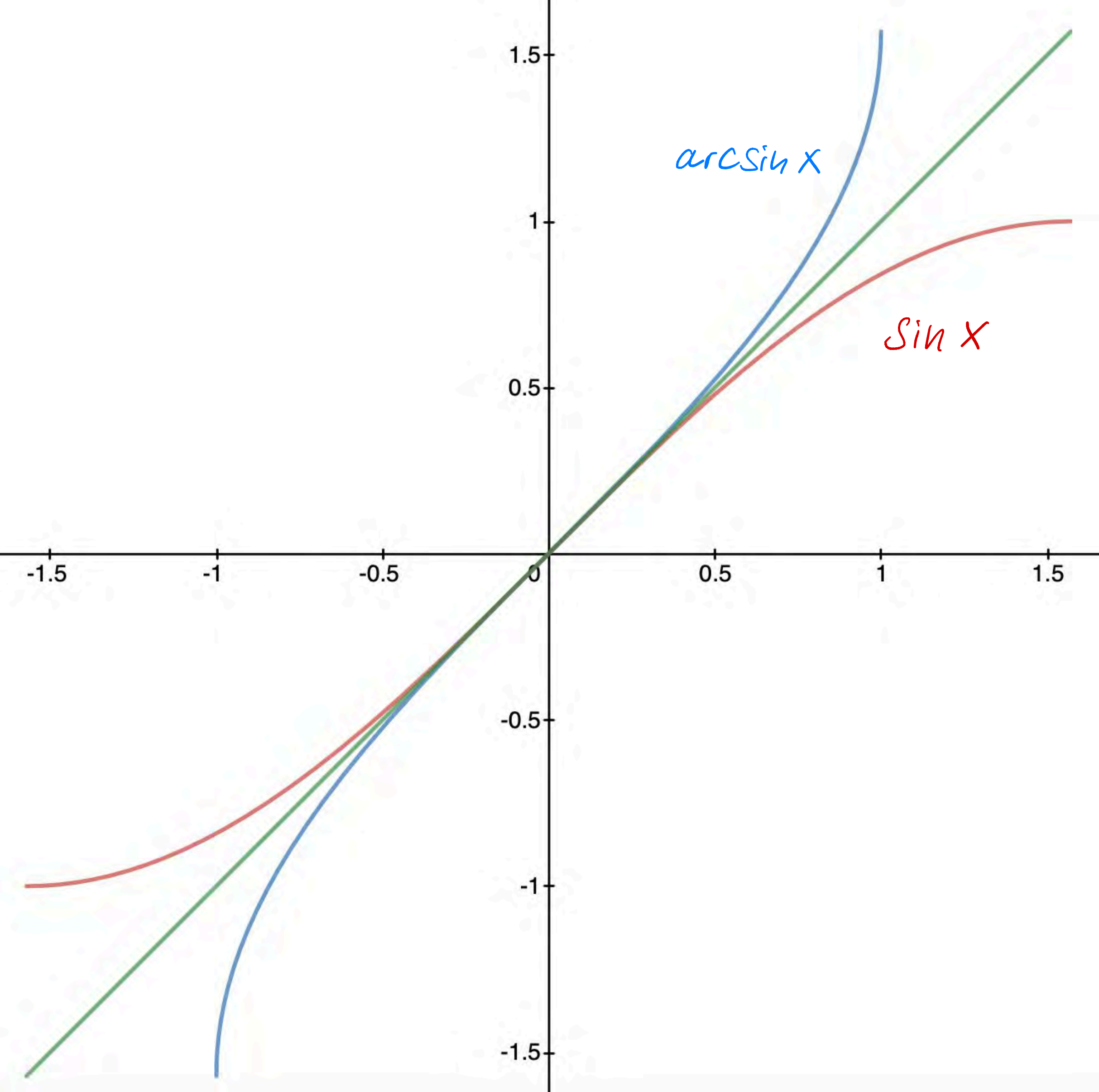

exp(x)
log(x)
3
2
1
0
-1
-2
-3
-4
4



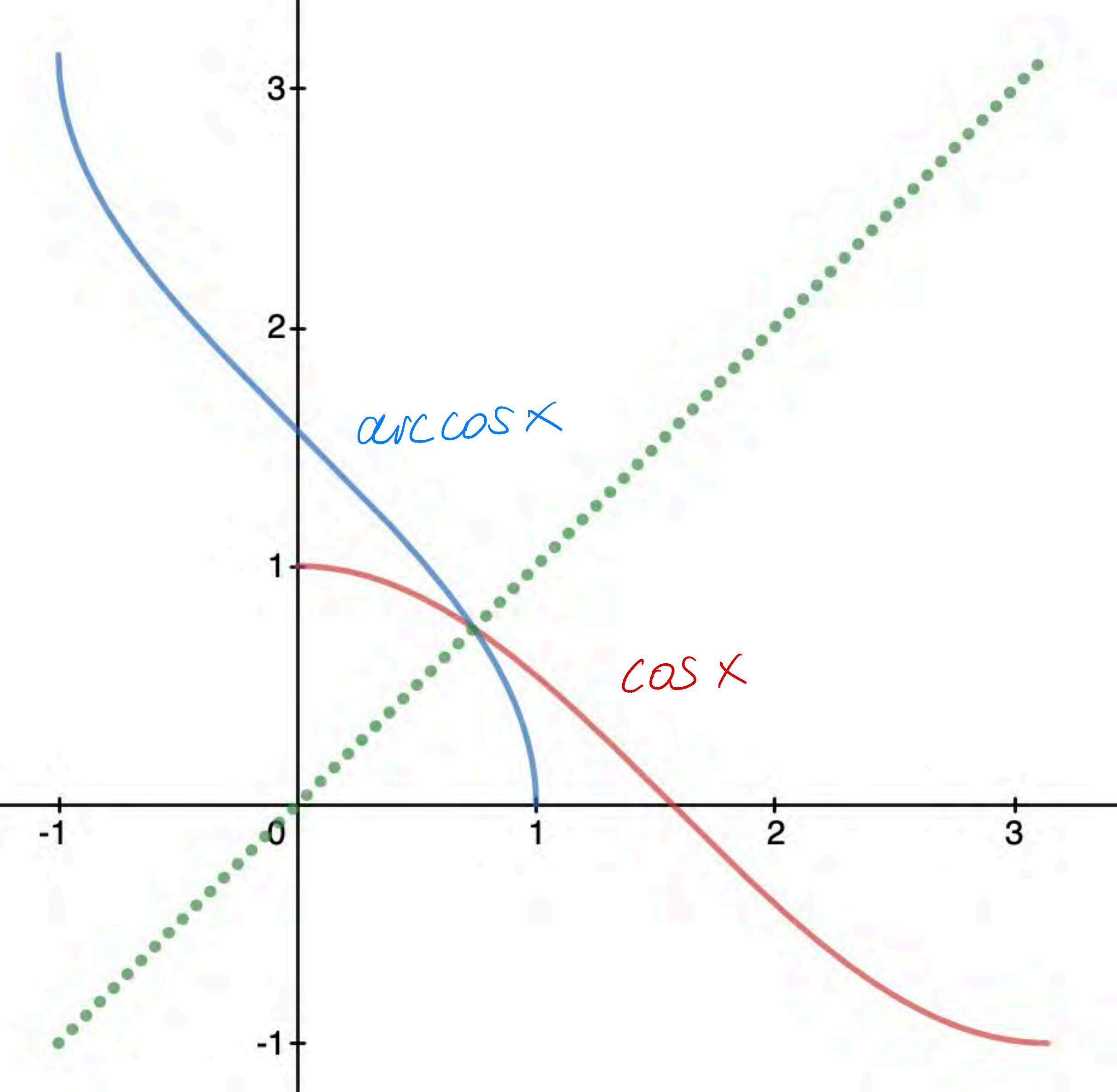
2.5
2
1.5
1
0.5
0
0.5
1
1.5
2
2.5
$x^2$
$\sqrt{x}$



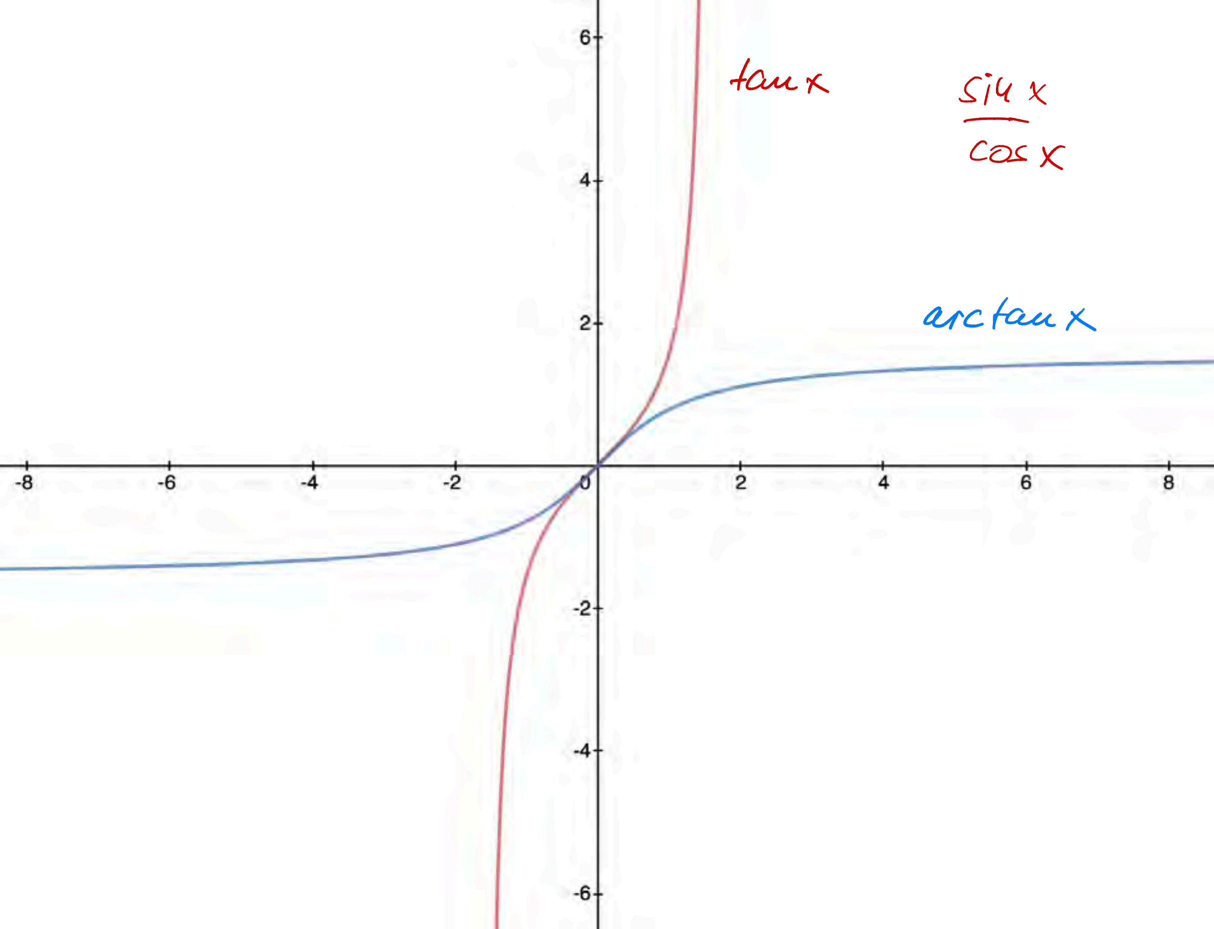
π/2
sin x
-4
-2
-π/2
0
2
4
0
cos x
π



1.5
1
0.5
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
arcsin x
sin x



3
2
1
-1
-1
0
1
2
3
arccos x
cos x


tan x
sin x / cos x
arctan x
6
4
2
-2
-4
-6
-8
-6
-4
-2
0
2
4
6
8

# Integrals of Inverse Functions

Suppose $g = f^{-1}$ is the inverse of $f$ and suppose $F' = f$.

What is $\displaystyle\int_a^b g(x)\,dx$ ?

We will have to see things that are not there :)

$$= \int_a^b \overset{u'}{1} \cdot \overset{v}{g(x)}\,dx$$

$$= \Big[ \overbrace{x}^{u} \cdot \overbrace{g(x)}^{v} \Big]_a^b - \int_a^b \overbrace{x}^{u} \cdot \overbrace{g'(x)}^{v'}\,dx$$

$$= \big[ x \cdot g(x) \big]_a^b - \int_a^b \overbrace{f}^{F'}(g(x)) \cdot g'(x)\,dx$$

$$= \big[ x \cdot g(x) \big]_a^b - \int_a^b \frac{d}{dx} F(g(x))\,dx$$

$$= \big[ x \cdot g(x) \big]_a^b - \big[ F(g(x)) \big]_a^b$$

This also shows that

$$x \cdot g(x) - F(g(x))$$

is an antiderivative of $g = f^{-1}$

# Examples: Integrals of Inverse Functions / 1

$$y \cdot g(y) - F(g(y)) = \int g(y)\,dy$$

antiderivative of $g$

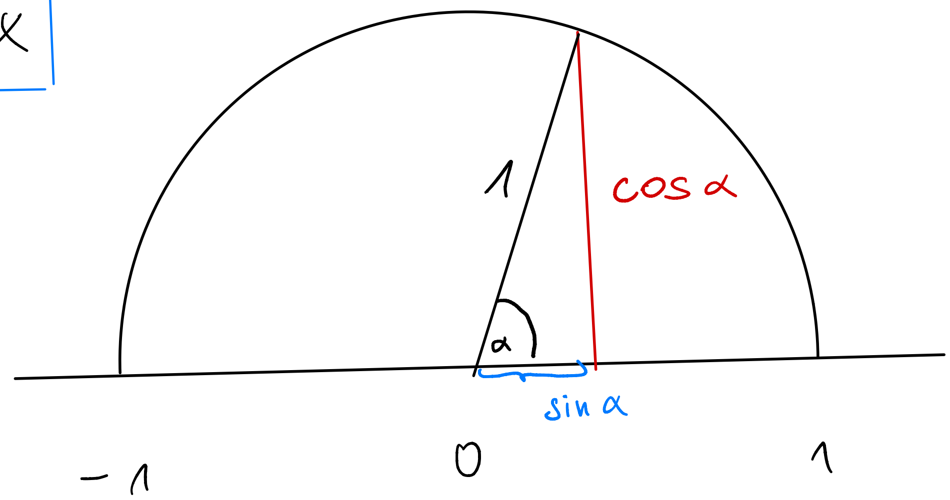
$$\boxed{g(x) = \log x}$$

$$\boxed{\int \log x \, dx} = x \cdot \log x - \exp(\log(x))$$

$$= \boxed{x \cdot \log x - x}$$

# Examples: Integrals of Inverse Functions / 2

$$\boxed{g(x) = \arcsin x}$$




We need a bit of background

- Pythagoras $\Rightarrow$ $\sin^2 + \cos^2 = 1$

$\Rightarrow$ $\cos^2 = 1 - \sin^2$

$\cos = \sqrt{1 - \sin^2}$

Now, $f = \sin$, $F = -\cos$

$$\int g(y)\,dy = y \cdot g(y) - F(g(y))$$

Hence

$$\boxed{\int \arcsin x \, dx} = x \cdot \arcsin x + \cos(\arcsin x))$$

$$= x \cdot \arcsin x + \sqrt{1 - \sin(\arcsin x)^2}$$

$$\boxed{= x \cdot \arcsin x + \sqrt{1 - x^2}}$$

$$\int \arcsin x \, dx = x \cdot \arcsin x + \sqrt{1-x^2}$$

Why is this plausible? Let's find the derivative of arcsin!

We have $\quad f = \sin, \quad g = \arcsin = f^{-1}$

$$f' = \cos = \sqrt{1-\sin^2}$$

By the inverse rule for derivation we get

$$g'(x) = \frac{1}{\underbrace{f'(g(x))}_{\cos(\arcsin(x))}} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}$$

Reading backwards, this gives us another antiderivative:

$$\boxed{\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x}$$

# The Derivative of arctan / 1

This will give us the antiderivative of an import function:

First, the derivative of tan:

$$\tan = \frac{\sin}{\cos} \implies \boxed{\tan'} = \frac{\sin' \cdot \cos - \sin \cdot \cos'}{\cos^2} = \frac{\cos \cdot \cos - \sin(-\sin)}{\cos^2}$$

$$= \frac{\cos^2 + \sin^2}{\cos^2} = \boxed{\frac{1}{\cos^2}}$$

$$\tan = \sin \cdot \frac{1}{\cos}$$

$$\frac{d}{dx}\frac{1}{\cos x} = -\frac{\cos' x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

# The Derivative of arctan /2

$f = \tan$, $\quad f' = \tan' = \dfrac{1}{\cos^2}$, $\quad g = \arctan$

this does not fit

$$\arctan' = \underbrace{\frac{1}{\tan' \circ \arctan}}_{\frac{1}{f' \circ f^{-1}}} \Rightarrow \arctan'(x) = \cos^2(\arctan(x))$$

$$\tan = \frac{\sin}{\cos} \Rightarrow \tan^2 = \frac{\sin^2}{\cos^2} = \frac{1-\cos^2}{\cos^2} = \frac{1}{\cos^2} - 1$$

$$\Rightarrow \frac{1}{\cos^2} = 1 + \tan^2 \Rightarrow \boxed{\cos^2 = \frac{1}{1+\tan^2}}$$

much better

$$\arctan' = \cos^2(\arctan(x)) = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}$$

We conclude

$$\boxed{\int \frac{1}{1+x^2}\,dx = \arctan x}$$

# Rules for indefinite integrals / 1

We have seen that the function

$$F(x) = \int_a^x f(y)\,dy \qquad (a \in \operatorname{dom} f)$$

is an antiderivative (= indefinite integral) of $f$.

We can therefore compute indefinite integrals as we did for definite integrals. It's actually easier:

- we can ignore the integral boundaries (any $a$ will do, and $x$ is anyway a variable)
- we can ignore the "boxes" around functions, i.e., we drop the $[\,\cdot\,]_a^b$

# Rules for Indefinite Integrals / 2

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int f'(x) \cdot g(x)\,dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\,dx$$

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(y)\,dy \Big|_{y=g(x)}$$

$$\int f^{-1}(x)\,dx = x \cdot f^{-1}(x) - \int f(y)\,dy \Big|_{y=f^{-1}(x)}$$

# Rules for Indefinite Integrals: Compact Form

$$\int c \cdot f = c \cdot \int f$$

$$\int f + g = \int f + \int g$$

$$\int f' \cdot g = fg - \int f \cdot g'$$

$$\int (f \circ g) \cdot g' = \left(\int f\right) \circ g$$

$$\int g = id \cdot g - \left(\int f\right) \circ g \quad , \quad \text{where } g = f^{-1}$$

# The Mechanics of Applying Substitution / 1

When calculating, thinking about $f$, $g$, and $g'$ is complicated. Instead, one uses Leibniz notation. Consider

$$\int x \cdot \cos x^2 \, dx$$

$$y = x^2$$

$$\frac{dy}{dx} = 2x \implies 2x \, dx = dy$$

$$= \int \frac{1}{2} \cdot \cos x^2 \cdot 2x \, dx$$

$$= \int \frac{1}{2} \cos y \, dy$$

$$= \frac{1}{2} \sin y$$

$$= \frac{1}{2} \sin x^2$$

This gives us also

$$\int_0^{\sqrt{\pi}} x \cdot \cos x^2 \, dx = \left[ \frac{1}{2} \sin x^2 \right]_0^{\sqrt{\pi}}$$

$$= \frac{1}{2} (\sin \pi - \sin 0) = 0$$

# The Mechanics of Applying Substitution / 1

When calculating, thinking about $f$, $g$, and $g'$ is complicated. Instead, one uses Leibniz notation. Consider

$$\int x \cdot \cos x^2 \, dx \quad = \quad F(5) - F(2)$$

$$= \int \cos(x^2) \cdot x \, dx$$

$$= \int \cos y \cdot \frac{1}{2} \, dy$$

$$= \frac{1}{2} \int \cos y \, dy$$

$$= \frac{1}{2} \sin y$$

$$= \frac{1}{2} \sin x^2 \quad = \quad F(x)$$

Substitution of $x^2$ by $y$:

$$y = x^2$$

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow \frac{1}{2} dy = x \cdot dx$$

# The Mechanics of Applying Substitution /2

We could also have taken the "inverse function approach"

$$\int x \cdot \cos x^2 \, dx$$

$$= \int \cancel{\sqrt{y}} \cdot \cos y \frac{1}{2\cancel{\sqrt{y}}} \, dy$$

$$= \int \frac{1}{2} \cos y \, dy$$

$$= \frac{1}{2} \sin y =$$

$$= \frac{1}{2} \sin x^2$$

$y = x^2 \Rightarrow x = \sqrt{y}$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$\Rightarrow dx = \frac{1}{2\sqrt{y}} \, dy$$

# The Mechanics of Applying Substitution / 3

Let's also redo our second example

$$y = \sqrt{x} \Rightarrow x = y^2$$

$$\Rightarrow \frac{dx}{dy} = 2y$$

$$\Rightarrow dx = 2y\, dy$$

$$\int \cos(\sqrt{x})\, dx$$

$$= \int \underbrace{\cos y}_{u'} \cdot \underbrace{2y}_{v}\, dy$$

$$= \sin y \cdot 2y - \int \underbrace{\sin y}_{u} \cdot \underbrace{2}_{v'}\, dy$$

$$= 2y \cdot \sin y - (-\cos y \cdot 2)$$

$$= 2y \cdot \sin y + 2 \cos y$$

$$= 2\sqrt{x} \cdot \sin\sqrt{x} + 2 \cos\sqrt{x}$$

$$\int \cos\sqrt{x}\,dx$$

$$= \int \cos y \cdot 2y\,dy$$

$$= 2\int \overset{u}{y}\;\overset{v'}{\cos y}\,dy$$

$$= 2\left( y\sin y - \int \overset{u'}{1}\;\overset{v}{\sin y}\,dy \right)$$

$$= 2y\cdot\sin y + \cos y$$

$$= 2\sqrt{x}\cdot\sin\sqrt{x} + \cos\sqrt{x}$$

$y = \sqrt{x}$

$\Rightarrow x = y^2$

$\Rightarrow \dfrac{dx}{dy} = 2y$

$\Rightarrow dx = 2y\,dy$

# Important Antiderivatives / Indefinite Integrals

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1}, \quad \alpha \neq -1$$

$$\int x^{-1}\,dx = \log x$$

$$\int e^{x}\,dx = e^{x}$$

$$\int \sin x\,dx = -\cos x \qquad \int \cos x\,dx = \sin x$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x$$

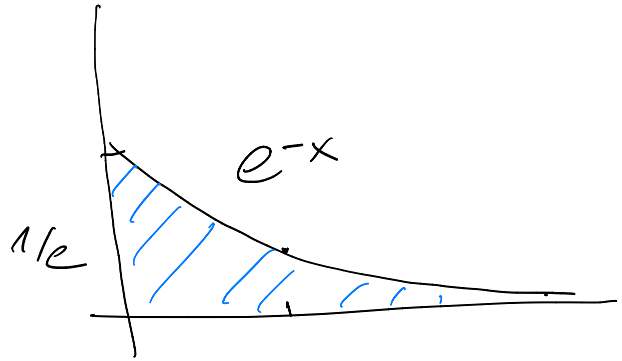
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x \qquad \int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x$$

Remember that antiderivatives are only unique up to a constant. Therefore, these are not proper equalities. Therefore, one often adds a "$+c$" to the end.

# Improper Integrals

How can we understand

$$\int_0^\infty e^{-x}\,dx \quad ?$$




For every $Y > 0$,

$$\int_0^Y e^{-x}\,dx = \left[-e^{-x}\right]_0^Y = -e^{-Y} - (-e^{-0}) = 1 - e^{-Y}$$

We note that

$$\lim_{Y\to\infty} 1 - e^{Y} = 0 .$$

We define $\displaystyle\int_0^\infty f(x)\,dx := \lim_{Y\to\infty} \int_0^Y f(x)\,dx$

and conclude that $\displaystyle\int_0^\infty e^{-x}\,dx = 1$

# Computing Integrals:

1. Symbolic integration: given a formula for $f$, find a formula for $\int f(x)\,dx$

2. Numerical integration: find the number $\int_a^b f(x)\,dx$

1) For every formula defining an "elementary" function $f$, we can compute a formula for $f'$.
This is different for integrals. E.g., $\int e^{-x^2}dx$ is not elementary.
There is an (extremely complicated) algorithm to compute integrals of elementary functions if they exist (Risch's algorithm, with 100 page description)
Computer algebra systems, like Wolfram Mathematica, use heuristics for symbolic integration.)

2) Methods exist since the late 17th century, based on approximation of areas. Newer methods are based on randomization and probabilistic techniques