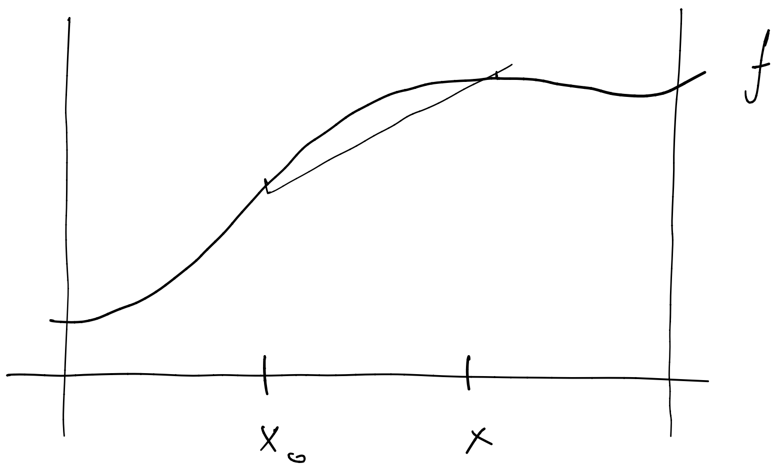
Derivatives and Integrals

Consider a function $f : [a,b] \longrightarrow \mathbb{R}$.

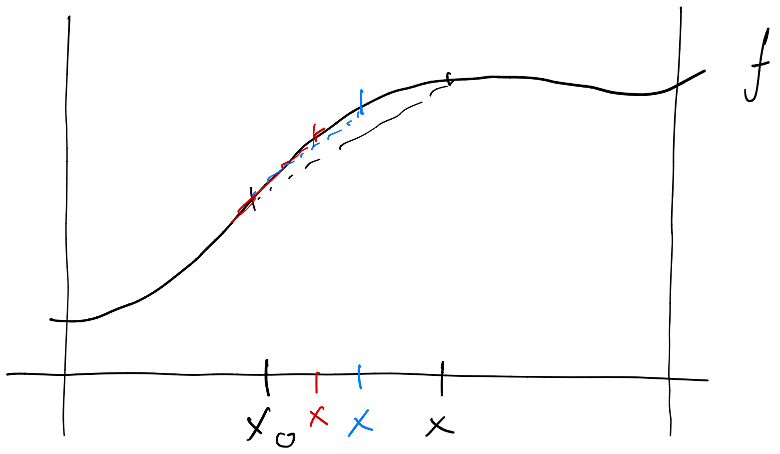
We want to define what is the steepness of the curve f



If $x_0 < x$, we can say what is the steepness over the stretch from x_0 to x: it is the gain in height divided by the length of the stretch:

$$\frac{f(x) - f(x_0)}{x - x_0}$$

What happens if we choose x ever closer to x_0? The result depends more and more on the immediate environment of x_0. If the limit of the quantity $\frac{f(x)-f(x_0)}{x-x_0}$ exists, then we can see it as the <u>steepness of f in position x_0</u>. We call it the derivative of f in x_0 and denote it as



$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Let's find derivatives of some simple functions.

1) $p_0(x) = 1$. Then

$$p_0'(x_0) = \lim_{x \to x_0} \frac{p_0(x_0) - p_0(x)}{x_0 - x} = \lim_{x \to x_0} \frac{1-1}{x-x_0} = \lim_{x \to x} 0 = 0$$

So, the constant function 1 has derivative 0.

2) $p_1(x) = x$. Then

$$p_1'(x_0) = \lim_{x \to x_0} \frac{p_1(x) - p_1(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1 = 1$$

3) $p_2(x) = x^2$. Then

$$p_2'(x_0) = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x + x_0)(x - x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} x + x_0 = 2x_0$$

That is

$$p_2'(x) = 2x$$

4) To deal with $p_n(x) = x^n$, we recall that

$$x^n - x_0^n = (x - x_0)(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \dots + x_0^{n-1}).$$ Then

$$\lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} (x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \dots + x_0^{n-1})$$

$$= n x_0^{n-1}$$

So, we have obtained that $p_n'(x) = n x^{n-1}$.

We introduce a new notation for derivatives. Instead of $f'(x)$ we write $\frac{d}{dx} f(x)$, indicating which letter is the variable according to which we are taking the derivative.*) Our past results can then be written as

$$\frac{d}{dx} 1 = 0, \quad \frac{d}{dx} x = 1, \quad \frac{d}{dx} x^2 = 2x, \quad \boxed{\frac{d}{dx} x^n = n x^{n-1}}$$

To obtain the derivative of x^n, we already had to use a special trick. To avoid tricks as far as possible, we derive a bunch of properties of derivatives.

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*) We call this also the Leibniz notation of derivatives

Multiplying f by a constant: consider $g(x) = c \cdot f(x)$, $c \in \mathbb{R}$

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c \cdot f(x) - c \cdot f(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} c \cdot \frac{(f(x) - f(x_0))}{x - x_0} = c \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = c \cdot f'(x_0)$$

Here we used that constants can be pulled out of a limit.
The calculation gives us the property

$$\boxed{(c \cdot f)' = c \cdot f',}$$

that is, multiplicative constants can be pulled out of the derivative.

Recall that limits are generally compatible with addition and multiplication:

$$\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

$$\lim_{x\to x_0} \left(f(x)\cdot g(x) \right) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$$

These properties are also known as "continuity of addition and multiplication.

We next apply the continuity of addition to derivatives.

Suppose $h(x) = f(x) + g(x)$. Then

$$h'(x_0) = \lim_{x\to x_0} \frac{f(x)+g(x) - (f(x_0)+g(x_0))}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{(f(x)-f(x_0)) + (g(x)-g(x_0))}{x-x_0}$$

continuity of $+$

$$= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}$$

$$= f'(x_0) + g'(x_0)$$

Hence $\boxed{(f+g)' = f' + g'}$

$$\lim_{x \to x_0} \left(f(x) \cdot g(x) \right) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

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We next apply the continuity of addition to derivatives.

Suppose $h(x) = f(x) + g(x)$. Then

$$h'(x_0) = \lim_{x \to x_0} \frac{f(x) + g(x) - (f(x_0) + g(x_0))}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}$$

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$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) + g'(x_0)$$

Hence $\boxed{(f+g)' = f' + g'}$

To derive a rule for <u>products</u>, we will use the trick of adding and subtracting a useful term (generally known as "17 camels trick").

Suppose $h(x) = f(x) \cdot g(x)$. Then

$$h'(x_0) = \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0) - g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} g(x)$$

$$+ f(x_0) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) \cdot g(x) + f(x_0) \cdot g'(x_0)$$

Hence, $\boxed{(f\cdot g)' = f'g + fg'.}$

In Leibniz notation, this is

$$\frac{d}{dx} f(x)\,g(x) = \left(\frac{d}{dx} f(x)\right) g(x) + f(x)\left(\frac{d}{dx} g x\right)$$

Applying the "product rule" to x^2 yields

$$\frac{d}{dx} x^2 = \frac{d}{dx} x\cdot x = \left(\frac{d}{dx}\cdot x\right)\cdot x + x\cdot\left(\frac{d}{dx} x\right)$$

$$= 1\cdot x + x\cdot 1 = 2x$$

We can also show by induction that $\frac{d}{dx} x^n = n x^{n-1}$:

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx} x\cdot x^n = \left(\frac{d}{dx} x\right)\cdot x^n + x\cdot\left(\frac{d}{dx} x^n\right)$$

$$= 1\cdot x^n + x\cdot n x^{n-1} = x^n + n x^n = (n+1) x^n$$

Chain Rule: We now consider a function that is the composition of two functions. Suppose $h(x) = f(g(x))$.

Then

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(g(x_0)) - f(g(x))}{x - x_0}$$

"17 camels trick"

$$= \lim_{x \to x_0} \frac{f(g(x_0)) - f(g(x))}{g(x_0) - g(x)} \cdot \frac{g(x_0) - g(x)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(g(x_0)) - f(g(x))}{g(x_0) - g(x)} \cdot \lim_{x \to x_0} \frac{g(x_0) - g(x)}{x - x_0}$$

If g is differentiable, then g is continuous; then $g(x) \to g(x_0)$ if $x \to x_0$.

$$= \lim_{y \to g(x_0)} \frac{f(g(x_0)) - f(y)}{g(x_0) - y} \cdot g'(x_0)$$

$$= f'(g(x_0)) \cdot g'(x_0)$$

This gives us the <u>chain rule</u>: $\boxed{(f \circ g)' = (f' \circ g) \cdot g'}$

Here, "$\circ$" is the composition operator for functions.

<u>Example:</u> Let us suppose we know that $\sin' = \cos$, that is, cosine is the derivative of sine.

What is the derivative of $\sin(x^2)$?

We can match the chain rule viewing f as $f(y) = \sin(y)$ and $g(x) = x^2$. Then $f'(y) = \cos(y)$ and $g'(x) = 2x$.

Then the chain rule tells us

$$\frac{d}{dx} \sin(x^2) = \sin'(x^2) \cdot 2x = \cos(x) \cdot 2x = 2x \cos(x)$$

We can remember the rule as telling us to first take the derivative of the outer function and then multiplying it with the one of the inner function: "outer derivative times inner derivative"

Inverse rule: Let g be the inverse function of f, that is, $g(f(y)) = y$ and $f(g(x)) = x$

For example, let $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be the exponential function $f(x) = \exp(x)$. Then f has an inverse function, the (natural) logarithm $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $g(y) = \log(y)$.

Note that not every function has an inverse. Note also that the existence of an inverse depends on what we assume to be domain and range of f.

Suppose, g is the inverse of f and let $h := f \circ g$ be the composition of f and g. Then $h(x) = x$. Hence

$$1 = \frac{d}{dx} x = \frac{d}{dx} h(x) = f'(g(x)) \cdot g'(x).$$

This gives us $\boxed{g'(x) = \frac{1}{f'(g(x))}}$

Example: $f(x) = \exp(x)$, $g(x) = \log x$

Let's assume we know that $\exp'(x) = \exp(x)$.

Actually, one way to define exp is to require that

(i) $\exp' = \exp$ (ii) $\exp(0) = 1$

Then there is exactly one function that satisfies these requirements.

What is $\log'$?

By the inverse rule

since $\exp' = \exp$

$$\boxed{\log'(x)} = \frac{1}{\exp'(\log x)} = \frac{1}{\exp(\log x)}$$

$$= \boxed{\frac{1}{x}}$$

since exp and log are inverses

Simple Quotient Rule: Suppose that $g(x) = \frac{1}{f(x)}$.

Consider $h(x) = \frac{f(x)}{f(x)} = f(x) \cdot g(x)$.

Then $h(x) = 1$. Hence $h'(x) = 0$. Therefore,

$$0 = h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\Rightarrow \quad f(x)\, g'(x) = -f'(x)\, g(x)$$

$$\Rightarrow \quad g'(x) = -\frac{f'(x)}{f(x)}\, g(x) = -\frac{f'(x)}{f(x)} \cdot \frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}$$

Hence,

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

Example: $f(x) = x^n$, $g(x) = x^{-n}$, $n \in \mathbb{N}$

That is, $g(x) = \dfrac{1}{f(x)}$. By the previous rule, we have

$$g'(x) = -\frac{f'(x)}{f^2(x)} = -\frac{n x^{n-1}}{(x^n)^2} = -n \frac{x^{n-1}}{x^{2n}}$$

$$= -n \frac{1}{x^{n+1}} = -n x^{-n-1}$$

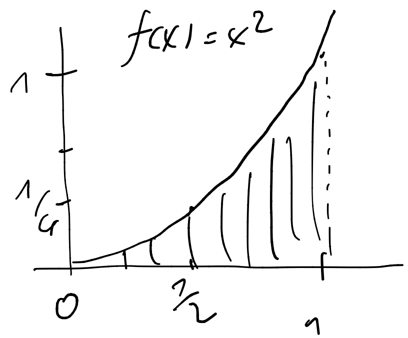
In short

$$\boxed{\frac{d}{dx} x^{-n} = -n x^{-n-1}}$$

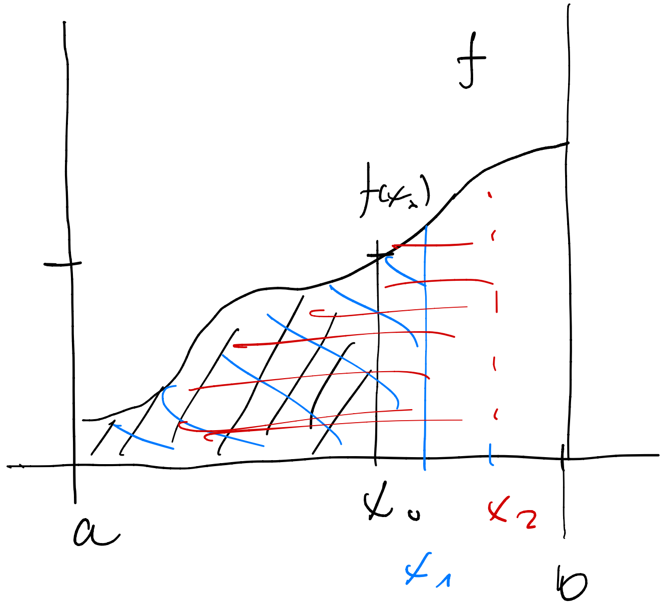
Area under the Curve

A question arising frequently in Geometry, Physics, and Engineering is to determine the area below the graph of a function.

Consider for example the function $f(x) = x^2$ and suppose we want to find out the area between the x-axis from 0 to 1 in our coordinate system and the graph of f:



We would like to do that for as general a class of functions as possible.



Let's suppose we have some "reasonable" function f (what exactly that means we will see as we develop our theory) on some interval $[a,b]$.

We generalize our problem to the one of finding the area under f from a to an arbitrary x.

Let's denote that area using a new symbol as

$$\int_a^x f(y)\,dy$$

and let the function F be defined as $F(x) := \int_a^x f(y)\,dy$.

If we know F, then we also know the area under f from some $x_0 \in [a,b]$ to some $x_1 \in [x_0, b]$: It is

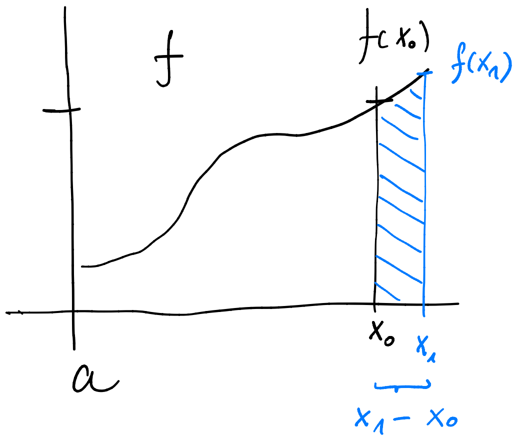
$$\int_{x_0}^{x_1} f(y)\,dy = \int_a^{x_1} f(y)\,dy - \int_a^{x_0} f(y)\,dy = F(x_1) - F(x_0)$$

Let us find out how f and F are connected!

We study how steep is F: from x_0 to x_1 the slope of f is

$$\frac{F(x_1) - F(x_0)}{x_1 - x_0}.$$

This has a geometric interpretation: It is the area of the blue column divided by the width $(x_1 - x_0)$.



Suppose that f is continuous. Then $f(x_1)$ gets ever closer to $f(x_0)$ as x_1 approaches x_0 and the blue area, which has size $F(x_1) - F(x_0)$, gets ever closer to the rectangle with height $f(x_0)$ and width $(x_1 - x_0)$, that is, $F(x_1) - F(x_0) \approx f(x_0)(x_1 - x_0)$.

and in the limit we have

$$\lim_{x_1 \to x_0} \frac{F(x_1) - F(x_0)}{x_1 - x_0} = f(x_0).$$

In summary, $F' = f$.

Question: How well do we know a function if we know its derivative?

Let G be another function with $G' = f$. Then

$(F-G)' = F' - G' = f - f = 0.$

What can we say about a function if its derivative is 0? We can conclude (by the mean-value theorem) that it is constant.*)

So suppose that $F(x) = \int_a^x f(y)\,dy$ and that we have a G such that $G' = f$. Then $(F-G)' = 0$, hence $F - G = c$ for some $c \in \mathbb{R}$, that is $G = F + c$.

We note that $F(a) = \int_a^a f(y)\,dy = 0$, since the area over an interval of length 0 is 0.

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* Of course, this needs an extra proof, which, however, is not too difficult.

We conclude:

$$\int_a^x f(y)\,dy = F(x) = F(x) - F(a)$$

"17 camels trick" $\rightsquigarrow$ $= F(x) + c - (F(a) + c) = G(x) - G(a)$

Consequences:

- If we want to know the area under f from a to b, we can find a function F such that $F' = f$. Then the area is

$$\int_a^b f(x)\,dx = F(a) - F(b)$$

- If we want to find a function G such that
  - $G' = f$
  - $G(a) = c$

and find $G(b)$ for some b, then we know $G(b) = c + \int_a^b f(x)\,dx$.

If we have a way to compute the area under f from a to b, then we can compute $G(b)$. Often, the area can be approximated.