2 Random Variables

We roll 2 dice, Interest: the sum $X = D_1 + D_2$.

$$X : S \longrightarrow \mathbb{R}$$

is a random variable

Idea: We are not interested in arbitrary events,
but only in the probabilities that a function X has
certain values,
eg., height/weight of people, marks of exams

$$P([X=2]) = \frac{1}{36}$$
$$P([X=3]) = \frac{2}{36}$$
$$P([X=4]) = \frac{3}{36}$$
$$P([X=5]) = \frac{4}{36}$$
$$P([X=6]) = \frac{5}{36}$$
$$P([X=7]) = \frac{6}{36}$$
$$P([X=8]) = \frac{5}{36}$$
$$P([X=9]) = \frac{4}{36}$$
$$P([X=10]) = \frac{3}{36}$$
$$P([X=11]) = \frac{2}{36}$$
$$P([X=12]) = \frac{1}{36}$$

Check $\sum_{i=2}^{12} P([X=i]) = 1$

A random variable

$$\mathcal{X} : \mathcal{S} \longrightarrow \mathbb{R}$$

is discrete if it has finitely or countably many values $x_1, x_2, \dots, x_n, \dots$

$\mathcal{X}$ is continuous if it takes a continuum of possible values (eg, weight, height)

Definition 24 The (cumulative) distribution function of $\mathcal{X}$ is

$$F : \mathbb{R} \to [0, 1]$$

$$F(x) := P[\mathcal{X} \leq x]$$

"$\mathcal{X} \sim F$" means "F is the distribution of $\mathcal{X}$"

F answers all probability questions about $\mathcal{X}$

Eg, $P[a < \mathcal{X} \leq b] = ?$

$[\mathcal{X} \leq b] = [\mathcal{X} \leq a] + [a < \mathcal{X} \leq b]$

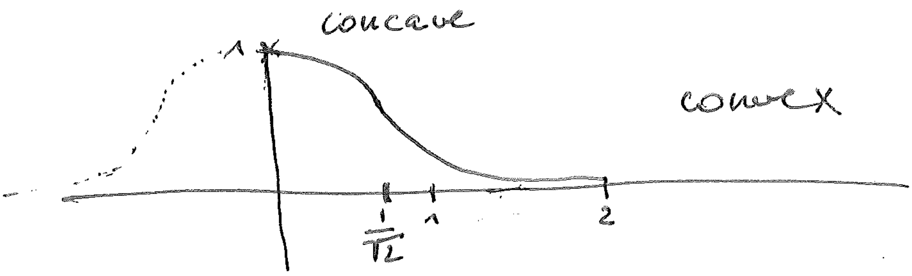
$$\Rightarrow P[a < \mathcal{X} \leq b] = P[\mathcal{X} \leq b] - P[\mathcal{X} \leq a]$$

$$= F(b) - F(a)$$

Example 25 : Suppose $\mathcal{X} \sim F$ where

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x^2} & x > 0 \end{cases}$$

Picture $G(x) = e^{-x^2}$



$G'(x) = -2x\, e^{-x^2} \Rightarrow$ G monotonically decreasing

$G''(x) = -2e^{-x^2} + (-2x)(-2x)\, e^{-x^2}$

$= (4x^2 - 2)\, e^{-x^2}$

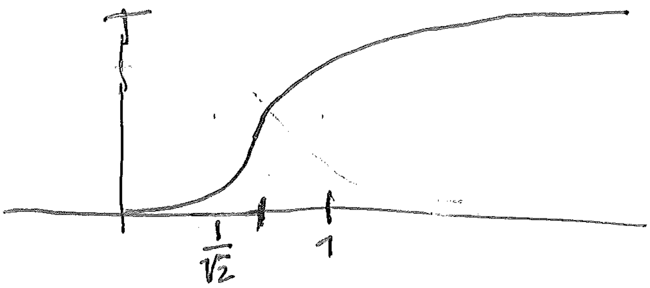
$G''(x) = 0 \iff 4x^2 = 2 \iff x^2 = \frac{1}{2} \iff x = \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} \approx 0.7$

$G''(x) \leq 0$ for $x < \frac{1}{\sqrt{2}}$ and ≥ 0 otherwise

$G(\frac{1}{\sqrt{2}}) = e^{-(\frac{1}{\sqrt{2}})^2} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

Picture $F(x)$:



$P[\mathcal{X} > 1] =$

$1 - P[\mathcal{X} \leq 1] =$

$1 - F(1) =$

$1 - (1 - e^{-1^2}) = e^{-1}$

$= 0.368$