Example: Consider a deck of French cards

$\mathcal{E}$ = draw a red card

$\mathcal{F}$ = draw an ace

$P(\mathcal{E}) = \frac{1}{2}$, $P(\mathcal{E}/\mathcal{F}) = \frac{1}{2}$

$P(\mathcal{F}) = \frac{1}{13}$, $P(\mathcal{F}/\mathcal{E}) = \frac{1}{13}$

We observe as well:

$P(\bar{\mathcal{E}}/\mathcal{F}) = \frac{1}{2}$

$P(\bar{\mathcal{F}}/\mathcal{E}) = \frac{12}{13}$

$P(\bar{\mathcal{E}}) = \frac{1}{2}$

$P(\bar{\mathcal{F}}) = \frac{12}{13}$

Symmetry:

$$\boxed{P(\mathcal{E}\mathcal{F}) = P(\mathcal{E})\,P(\mathcal{F})}$$

Definition of independence of events

$$\Leftrightarrow P(\mathcal{E}/\mathcal{F}) = P(\mathcal{E})$$

$$\Leftrightarrow P(\mathcal{F}/\mathcal{E}) = P(\mathcal{F})$$

if $P(\mathcal{E}) \neq 0 \neq P(\mathcal{F})$

Proposition 20 $\quad \mathcal{E}, \mathcal{F}$ independent $\Rightarrow$ $\mathcal{E}, \overline{\mathcal{F}}$ independent

Proof: Show $\quad P(\mathcal{E}\overline{\mathcal{F}}) = P(\mathcal{E})P(\overline{\mathcal{F}})$

$$P(\mathcal{E}) = P(\mathcal{E}\mathcal{F} \cup \mathcal{E}\overline{\mathcal{F}}) = P(\mathcal{E}\mathcal{F}) + P(\mathcal{E}\overline{\mathcal{F}})$$

$$= P(\mathcal{E}) \cdot P(\mathcal{F}) + P(\mathcal{E}\overline{\mathcal{F}})$$

$$\Rightarrow \quad \underbrace{P(\mathcal{E}) - P(\mathcal{E}) \cdot P(\mathcal{F})}_{=\,(1 - P(\mathcal{F}))P(\mathcal{E})} = \underline{P(\mathcal{E}\overline{\mathcal{F}})}$$

$$(1 - P(\mathcal{F}))P(\mathcal{E}) = \underline{P(\overline{\mathcal{F}}) \cdot P(\mathcal{E})}$$

$\mathcal{E}, \mathcal{F}$ ind. $\Rightarrow$ $\overline{\mathcal{E}}, \overline{\mathcal{F}}$ ind.

Independence is inherited by complements

Consider $\mathcal{E}$ ind of $\mathcal{F}$, $\quad \not\Longrightarrow \quad \mathcal{E}$ ind $\mathcal{F} \cap \mathcal{G}$
$\mathcal{E}$ ind of $\mathcal{G}$

<u>Example:</u> Throw two dice

$\mathcal{E} =$ "$D_1 + D_2 = 7$" $\qquad \mathcal{F} =$ "$D_1 = 1$" $\qquad \mathcal{G} =$ "$D_2 = 6$"

$P(\mathcal{E}) = \frac{1}{6} \qquad P(\mathcal{F}) = \frac{1}{6} \qquad P(\mathcal{G}) = \frac{1}{6}$

$P(\mathcal{E}\mathcal{F}) = \frac{1}{36} \qquad P(\mathcal{E}\mathcal{G}) = \frac{1}{36} \qquad P(\mathcal{F}\mathcal{G}) = \frac{1}{36}$

$\Longrightarrow \mathcal{E}, \mathcal{F}$ ind. $\qquad \mathcal{E}, \mathcal{G}$ ind

What about ind. of $\mathcal{E}$ and $\mathcal{F} \cap \mathcal{G} = \mathcal{F}\mathcal{G}$?

$$P(\mathcal{E})P(\mathcal{F}\mathcal{G}) = \frac{1}{6}\frac{1}{36} \neq \frac{1}{36} = P(\mathcal{E}\mathcal{F}\mathcal{G}) = P(\mathcal{E}(\mathcal{F}\mathcal{G}))$$

$\Longrightarrow \mathcal{E}$ and $\mathcal{F}\mathcal{G}$ are <u>not</u> ind.

Definition 22: $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are *independent* if

- $\mathcal{E}, \mathcal{F}$ and $\mathcal{E}, \mathcal{G}$ and $\mathcal{F}, \mathcal{G}$ are ind. (pairw. ind.)
- $P(\mathcal{E}\mathcal{F}\mathcal{G}) = P(\mathcal{E})P(\mathcal{F})P(\mathcal{G})$

Now, $\mathcal{E}$ and $\mathcal{F}\mathcal{G}$ are ind.

$$P(\mathcal{E}\cdot(\mathcal{F}\mathcal{G})) = P(\mathcal{E}\mathcal{F}\mathcal{G}) = P(\mathcal{E})\underbrace{P(\mathcal{F})\ P(\mathcal{G})}$$

$$= P(\mathcal{E})\ P(\mathcal{F}\mathcal{G})$$

$\leftarrow$ ind. of $\mathcal{F}, \mathcal{G}$

Remark: $\mathcal{E}, \mathcal{F}, \mathcal{G}$ ind $\Rightarrow$ $\mathcal{E}$ and $(\mathcal{F} \cup \mathcal{G})$ ind.

$$
\begin{aligned}
P(\mathcal{E}(\mathcal{F} \cup \mathcal{G})) &= P(\mathcal{E}\mathcal{F} \cup \mathcal{E}\mathcal{G}) \\
&= P(\mathcal{E}\mathcal{F}) + P(\mathcal{E}\mathcal{G}) - P(\mathcal{E}\mathcal{F}\mathcal{G}) \\
&= P(\mathcal{E})P(\mathcal{F}) + P(\mathcal{E})P(\mathcal{G}) - P(\mathcal{E})\overbrace{P(\mathcal{F})P(\mathcal{G})}^{=P(\mathcal{F}\mathcal{G})} \\
&= P(\mathcal{E})\big(P(\mathcal{F}) + P(\mathcal{G}) - P(\mathcal{F}\mathcal{G})\big) \\
&= P(\mathcal{E})\, P(\mathcal{F} \cup \mathcal{G})
\end{aligned}
$$

Generalized Definition $E_1, \ldots, E_n$ are <u>independent iff</u>

for every subset of $F_1, \ldots, F_m$:

$$P(F_1 \cdots F_m) = P(F_1) \cdots P(F_m)$$

<u>Examples</u>: • Sequences of experiments; E_i refers to i-th execution of experiment

E.g.: Rolling die

• Sequences of disease tests:

Intuition for one test being positive for people w/ and w/o disease:

$$P(T \mid D) = .99 \qquad P(\bar{T} \mid \bar{D}) = .99$$

$$P(T \mid \bar{D}) = .01$$

prob. of false positives

- Sequences of disease tests

Example probabilities for a test being positive for people w/ and w/o disease

specificity

$$P(T \mid D) = .99 \qquad\qquad P(\bar{T} \mid \bar{D}) = .99$$

sensitivity

$$\Leftrightarrow P(T \mid \bar{D}) = .01$$

If $P(D)$ is low (e.g., 1%), then we cannot rely on the test because there are as many true positive as false positive test results.

- What can we do?

Idea: Apply the test twice! First T_1, then T_2.

But: Need to ensure that probabilities multiply!

That is

$$P(T_1 T_2 \mid D) = P(T_1 \mid D)\, P(T_2 \mid D)$$

numbers from our example

$$= \frac{99}{100} \cdot \frac{99}{100} = \frac{9801}{10\,000} \approx 98\%$$

This property of T_1, T_2, D is spelt out as

"T_1 and T_2 are conditionally independent given D"

It is independence of T_1, T_2 with regard to the probability measure

$$P(\cdot \mid D)$$

remember that for every probability $P(\cdot)$ on S also $P(\cdot \mid D)$ is a probability on S.

Assume that T_1, T_2 are also independent given $\bar{D}$.

Then

$$P(T_1, T_2 \mid \bar{D}) = P(T_1 \mid \bar{D}) \cdot P(T_2 \mid \bar{D})$$

numbers from our example

$$= \frac{1}{100} \cdot \frac{1}{100} = \frac{1}{10\,000}$$

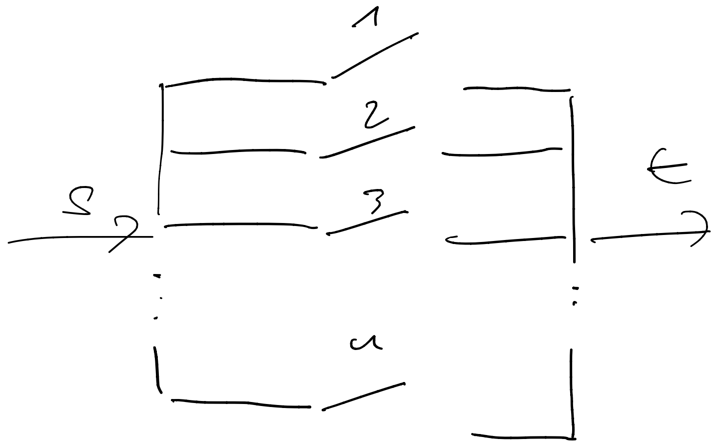
This shows that the probability of false positives has been sharply reduced: The relationship between false and true positives now is

numbers from our example

$$\frac{1}{9801}$$

A Bayesian analysis of what can be concluded from the other test results ($T_1 \bar{T}_2$, $\bar{T}_1 T_2$ and $\bar{T}_1 \bar{T}_2$) will be part of the assignment.

Example 23



components are ind. work with p_i for comp i

System works if ≥ 1 comp. works

E = "system works", F_i = "comp i works" $\Rightarrow P(E) = ?$

$\overline{E}$ = "system doesn't work" if no comp. works

if $\overline{F_1}$ and $\overline{F_2}$ ... $\overline{F_n}$

Note: (F_i) ind $\Rightarrow$ $(\overline{F_i})$ ind

$$P(E) = 1 - P(\overline{E}) = 1 - P(\overline{F_1} \cdots \overline{F_n}) = 1 - P(\overline{F_1}) \cdots P(\overline{F_n})$$

$$= 1 - \prod_{i=1}^{n} P(\overline{F_i}) = 1 - \prod_{i=1}^{n} (1 - P(F_i)) = 1 - \prod_{i=1}^{n} (1 - p_i)$$