General Law of Total Probability (LOTP)

Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be a partition of $\mathcal{S}$, i.e.

- $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset, \quad i \neq j$
- $\bigcup_{i=1}^{n} \mathcal{F}_i = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_n = \mathcal{S}$

Note:

$$\sum_{i=1}^{n} P(\mathcal{F}_i) = 1$$

$\mathcal{E} \subseteq \mathcal{S}$ event $\Longrightarrow \mathcal{E} = \biguplus_{i=1}^{n} \mathcal{E}\mathcal{F}_i$

$$\Rightarrow P(\mathcal{E}) = \sum_{i=1}^{n} P(\mathcal{E}\mathcal{F}_i) = \sum_{i=1}^{n} P(\mathcal{E} \mid \mathcal{F}_i)\, P(\mathcal{F}_i)$$

Quiz 11: Marbles

LOTP: What are $\mathcal{E}$ and the $\mathcal{F}_i$?

$\mathcal{E}$ = Marble drawn is red

$\mathcal{F}_i$ = The i-th bag is chosen

$$P(\mathcal{E}) = P(\mathcal{E} \mid \mathcal{F}_1) P(\mathcal{F}_1) + P(\mathcal{E} \mid \mathcal{F}_2) P(\mathcal{F}_2) + P(\mathcal{E} \mid \mathcal{F}_3) P(\mathcal{F}_3)$$

$$P(\mathcal{F}_1) = P(\mathcal{F}_2) = P(\mathcal{F}_3) = \frac{1}{3}$$

$$P(\mathcal{E} \mid \mathcal{F}_1) = \frac{80}{100} \qquad P(\mathcal{E} \mid \mathcal{F}_2) = \frac{55}{100} \qquad P(\mathcal{E} \mid \mathcal{F}_3) = \frac{45}{100}$$

$$\Rightarrow \quad P(\mathcal{E}) = \frac{1}{3} \, \frac{80 + 55 + 45}{100} = \frac{1}{\cancel{3}_{1}} \, \frac{\cancel{180}^{60}}{100} = \frac{60}{100}$$

$(F_i)_{i=1}^{n}$ partition of S, E event

$$P(F_i \mid E) = $$

$$P(F_i \mid E) = \frac{P(F_i E)}{P(E)} = \frac{P(E F_i)}{P(E)}$$

$$= \frac{P(E \mid F_i) P(F_i)}{P(E)}$$

$$= \frac{P(E \mid F_i) P(F_i)}{\sum_{i=1}^{n} P(E \mid F_i) P(F_i)}$$

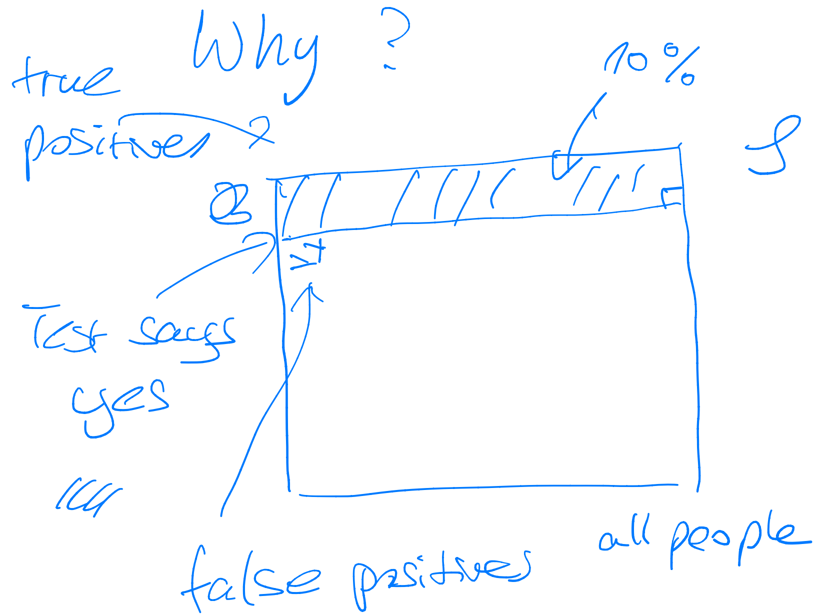
Generalized Bayes' Formula

Quiz 10: Bivid-20 Diagnosis

B person has Bivid

T test is positive

Interesting: $P(B \mid T) = ?$



$$P(T \mid B) = \frac{99}{100}$$

$$P(\bar{T} \mid \bar{B}) = \frac{99}{100} \Rightarrow P(T \mid \bar{B}) = \frac{1}{100}$$

$$P(B) = \frac{10}{100}$$

$$P(B \mid T) = \frac{P(T \mid B)\, P(B)}{P(T)} = \frac{\frac{99}{\cancel{100}} \cdot \frac{\cancel{10}}{\cancel{100}}}{\frac{99}{\cancel{100}} \cdot \frac{\cancel{10}}{\cancel{100}} + \frac{1}{\cancel{100}} \cdot \frac{\cancel{90}}{\cancel{100}}}$$

$$P(T) = P(T \mid B)\, P(B) + P(T \mid \bar{B}) \cdot P(\bar{B})$$

$$= \frac{99}{99 + 9} = \frac{99}{108} = 91.67 \approx 92$$

1.6 Independent Events

Example: Consider a deck of French cards

$\mathcal{E}$ = draw a red card

$\mathcal{F}$ = draw an ace

$P(\mathcal{F}) = \frac{4}{52} = \frac{1}{13}$

$P(\mathcal{F} | \mathcal{E}) = \frac{2}{26}$

red aces

red cards

$P(\mathcal{E}) = \frac{1}{2}$, $\quad P(\mathcal{E} | \mathcal{F}) = \frac{1}{2}$

Intuition: Knowing $\mathcal{F}$ doesn't tell us anything about $\mathcal{E}$

$\mathcal{E}, \mathcal{F}$ are independent

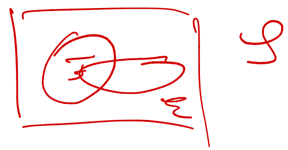
In general, $P(\mathcal{E} | \mathcal{F}) \neq P(\mathcal{E})$

prior probality of $\mathcal{E}$

posterior prob. of $\mathcal{E}$

Definition: $\mathcal{E}, \mathcal{F}$ independent $\quad P(\mathcal{E} | \mathcal{F}) = P(\mathcal{E})$

Note: the definition is symmetric



$$P(\mathcal{E} \mid \mathcal{F}) = P(\mathcal{E}) \iff P(\mathcal{E}) = \frac{P(\mathcal{E}\mathcal{F})}{P(\mathcal{F})}$$

if $P(\mathcal{F}) \neq 0$, $P(\mathcal{E}) \neq 0$

$$\iff \boxed{P(\mathcal{E})P(\mathcal{F}) = P(\mathcal{E}\mathcal{F})}$$

The first definition assumes that $P(\mathcal{F}) \neq 0$

Alternative:

$\mathcal{E}, \mathcal{F}$ indep. iff $P(\mathcal{E}\mathcal{F}) = P(\mathcal{E}) \cdot P(\mathcal{F})$

$$P(\mathcal{E})P(\mathcal{F}) = P(\mathcal{E}\mathcal{F}) \Rightarrow P(\mathcal{F}) = \frac{P(\mathcal{E}\mathcal{F})}{P(\mathcal{E})}$$

$$= \frac{P(\mathcal{F}\mathcal{E})}{P(\mathcal{E})} = P(\mathcal{F} \mid \mathcal{E})$$

Quiz: Dice and Independence

E = "$D_1 + D_2 = 7$"

F = "$D_1 + D_2 = 8$"

G = "$D_1 = 3$"

$F = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$

$\Rightarrow \# F = 5$

$G = \{(3,1), \dots, (3,6)\}$

$\Rightarrow \# G = 6$

E ind F : $P(EF) = 0 \neq \frac{1}{6} \cdot \frac{5}{36} = P(E)P(F)$ No!

E ind G : $P(EG) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(E)P(G)$ Yes!

F ind G : $P(FG) = \frac{1}{36} \neq \frac{5}{36} \cdot \frac{1}{6} = P(F)P(G)$ No!

$P(E) = \frac{6}{36} = \frac{1}{6}$

$P(F) = \frac{5}{36}$

$P(G) = \frac{6}{36} = \frac{1}{6}$

$P(EF) = 0$

$P(EG) = \frac{1}{36}$

$P(FG) = \frac{1}{36}$