<u>Generalisation.</u> Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be a <u>partition</u> of $\mathcal{S}$, i.e.,

- $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset, \quad i \neq j$
- $\bigcup_{i=1}^{n} \mathcal{F}_i = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_n = \mathcal{S}$.

$\mathcal{E}$ event $\Rightarrow \quad \mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}\mathcal{F}_i$

$$\Rightarrow P(\mathcal{E}) = \sum_{i=1}^{n} P(\mathcal{E}\mathcal{F}_i) = \sum_{i=1}^{n} P(\mathcal{E}|\mathcal{F}_i)P(\mathcal{F}_i)$$

Previously, $\mathcal{F}, \bar{\mathcal{F}}$ formed a partition.

Assume $\mathcal{E}$ happened.

$$P(\mathcal{F}_i|\mathcal{E}) = \frac{P(\mathcal{E}|\mathcal{F}_i)\,P(\mathcal{F}_i)}{P(\mathcal{E})}$$

$$= \frac{P(\mathcal{E}|\mathcal{F}_i)\,P(\mathcal{F}_i)}{\sum_{j=1}^{n} P(\mathcal{E}|\mathcal{F}_j)\,P(\mathcal{F}_j)}$$

Describes how opinion about $\mathcal{F}_i$ changes in presence of $\mathcal{E}$.

Example A new test for the early diagnosis of cancer has been developed with the following characteristics:

Events: $T$ test positive $C$ person has cancer

$$P(T|C) = \frac{99}{100} \qquad P(T|\bar{C}) = \frac{1}{100}$$

$$P(C) = \frac{1}{100}$$

What is $P(C|T)$ ?

$$P(C|T) = \frac{P(T|C)P(C)}{P(T)}$$

$$= \frac{P(T|C)\,P(C)}{P(T|C)P(C) + P(T|\bar{C})P(\bar{C})}$$

$$= \frac{\frac{99 \cdot 1}{100 \cdot 100}}{\frac{99 \cdot 1 \quad + \quad 1 \cdot 99}{100 \cdot 100}}$$

$$= \frac{1}{2}$$

That is: In 50% of all cases the test result is wrong

Note: To judge the quality of a test, we need to know not only $P(T|C)$, but also $P(T|\bar{C})$ (= false positives) and $P(C)$

Example 17 Suppose program $P$ has 3 parts, $P_1, P_2, P_3$. $P$ has a bug, which may be in any $P_i$ with probability $\frac{1}{3}$.

Let $1-\alpha_i$ be the probability of finding the bug in $P_i$ if it is there.

What is the probability that the bug is in $P_1, P_2, P_3$ if we did not find it in $P_1$?

- $\mathcal{B}_i$ bug in part $P_i$
- $\mathcal{F}_1$ find bug in $P_1$.

We want $P(\mathcal{B}_i \mid \overline{\mathcal{F}_1})$

$$P(\mathcal{B}_i \mid \overline{\mathcal{F}_1}) = \frac{P(\overline{\mathcal{F}_1} \mid \mathcal{B}_i) \cdot P(\mathcal{B}_i)}{P(\overline{\mathcal{F}_1})}$$

$$P(\overline{\mathcal{F}_1}) = P(\overline{\mathcal{F}_1} \mid \mathcal{B}_1) P(\mathcal{B}_1) + P(\overline{\mathcal{F}_1} \mid \mathcal{B}_2) P(\mathcal{B}_2) + P(\overline{\mathcal{F}_1} \mid \mathcal{B}_3) P(\mathcal{B}_3)$$

$$= \alpha_1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = (\alpha_1 + 2)\frac{1}{3}$$

$$P(\overline{\mathcal{F}_1} \mid \mathcal{B}_1) \cdot P(\mathcal{B}_1) = \alpha_1 \cdot \frac{1}{3}$$

$$P(\overline{\mathcal{F}_1} \mid \mathcal{B}_i) \cdot P(\mathcal{B}_i) = 1 \cdot \frac{1}{3} \qquad i = 1, 2$$

Hence

$$P(\mathcal{B}_1 \mid \overline{\mathcal{F}_1}) = \frac{\alpha_1 \cdot \frac{1}{3}}{(\alpha_1 + 2)\frac{1}{3}} = \frac{\alpha_1}{\alpha_1 + 2}$$

$$P(\mathcal{B}_i \mid \overline{\mathcal{F}_1}) = \frac{1 \cdot \frac{1}{3}}{(\alpha_1 + 2)\frac{1}{3}} = \frac{1}{\alpha_1 + 2}$$

## 1.6 Independent Events

In general, $P(E|F) \neq P(E)$

**Definition 18** $E, F$ are independent if
$P(E|F) = P(E)$. *)

**Example 19** Consider a deck of French cards.

$E$ = draw a red card

$F$ = draw an ace

$P(E) = \frac{1}{2}$, $P(E|F) = \frac{1}{2}$ $\Rightarrow$ $E, F$ indep.

**Proposition 20**

$E, F$ independent $\Rightarrow$ $E, \bar{F}$ independent

Proof: Show $P(\bar{F}|E) = P(\bar{F})$

$$P(\bar{F}|E) = 1 - P(F|E) = 1 - \frac{P(FE)}{P(E)}$$

$$= 1 - \frac{P(F)P(E)}{P(E)} = 1 - P(F) = P(\bar{F}) \quad \square$$

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* Note: the definition is symmetric

$$P(E|F) = P(E) \Rightarrow P(E) = \frac{P(EF)}{P(F)}$$

$$\Rightarrow P(EF) = P(E)P(F)$$

$$\Rightarrow P(F|E) = P(F)$$