Quiz 3: Socks in a Box

$$P(\underbrace{\text{2 socks have diff. colour}}_{\mathcal{E}}) = \frac{\# \mathcal{E}}{\# \mathcal{S}}$$

8 black, 7 white

$\mathcal{S}$ = Pairs of socks picked, first, then second $\Rightarrow \# \mathcal{S} = 15 \times 14$

$\mathcal{E}$ = Pairs with $(1st = w \wedge 2nd = b) \cup (1st = b \wedge 2nd = w)$

$\# \mathcal{E} \qquad 7 \times 8 + 8 \times 7$

$$\frac{\# \mathcal{E}}{\# \mathcal{S}} = \frac{7 \cdot 8 + 8 \cdot 7}{15 \cdot 14} = \frac{\cancel{2} \cdot \cancel{7} \cdot 8}{15 \cdot \cancel{14}} = \frac{8}{15}$$

Quiz 4: How many words?

\# Words = # possibilities 1st bit

× # — " — 2nd — " —

. . .

× # — " — 32nd ~~bit~~

$2 \times 2 \times \dots \times 2$

$= 2^{32}$ // $2^{10} = 1024 \approx 1000 = 10^3$

$= 2^{30} \cdot 2^2 = (2^{10})^3 \cdot 2^2$

$\approx (10^3)^3 \cdot 2^2 = 4{,}000{,}000{,}000$

Quiz 5: People on an Elevator

$$P(\underbrace{\text{"all off at same floor"}}_{\mathcal{E}})$$

$$\mathcal{E}_i = \text{"all off at floor } i\text{"} \qquad (i = 1, \dots, 4)$$

$$\mathcal{S} = \{ (f_1, f_2, f_3, f_4) \mid f_i \in \{1,2,3,4\},\ i = 1, \dots, 4 \}$$

$$= \{ (1,1,1,1), (1,1,1,2), \dots \}$$

$\#\mathcal{S} = 4^4$ // In general: $f = \#$floors, $p = \#$persons

f^p

$$\mathcal{E}_i = \{(i,i,i,i)\} \Rightarrow \#\mathcal{E}_i = 1$$

$$\mathcal{E} = \biguplus_{i=1}^{f} \mathcal{E}_i \Rightarrow \#\mathcal{E} = f$$

$$\frac{\#\mathcal{E}}{\#\mathcal{S}} = \frac{\cancel{f^1}\ 1}{f^{\cancel{p}}\ p-1} = \frac{1}{f^{p-1}}.$$

Here: $\dfrac{\#\mathcal{E}}{\#\mathcal{S}} = \dfrac{1}{4^3} = \dfrac{1}{2^6}$

$$= \frac{1}{64}$$

disjoint union,
i.e., the $\mathcal{E}_i$ are mut. disjoint

Example 6 10 books: 4 CS, 3 Math, 2 stat, 1 Hist

Organize so that book of same subject are together:

E.g., SS . CCCC . H . MMM

How many possibilities?

1) # permutations of subjects $4 \cdot 3 \cdot 2 \cdot 1 = 4!$

2) permutation within subjects, e.g. CS : $4!$

M : $3!$

S : $2!$

H : $1!$

$\# \text{Arrangements} = 4! \cdot 4! \cdot 3! \, 2! \, 1!$

$= (2^3 \cdot 3) \cdot (2^3 \cdot 3) \cdot (2^1 \cdot 3) \cdot 2^1$

$= 2^8 \cdot 3^3 = 256 \cdot 27$

$\approx 250 \cdot 28$

$= 250 \cdot 4 \cdot 7 = 7 \cdot 1000 = 7000$

$4! = 4 \cdot 3 \cdot 2$

$= 2^2 \cdot 3 \cdot 2$

$= 2^3 \cdot 3$

Arrangement, w/ book mixed up arbitrarily

$10 \cdot 9 \cdot 8 \cdots 1 = 10!$

Random arrang. of book leading to subjects together

$\approx \frac{7000}{10!}$

Example 7   Course with 5 male, 3 female students.

We had an exam: all students got different mark

$$P(\underbrace{\text{"all female students got the top marks"}}_{\mathcal{E}}) =$$

$\mathcal{S}$ = all possible rankings, $\#\mathcal{S} = 8!$

Females Top Rankings:

$$\underset{3!}{\boxed{\;F\;}}\ \underset{5!}{\boxed{\;M\;}}$$

$$P(\mathcal{E}) = \frac{\#\mathcal{E}}{\#\mathcal{S}} = \frac{3!\,5!}{8!} = \frac{3!\,\cancel{5!}}{8\cdot 7\cdot 6\cdot \cancel{5!}} = \frac{\cancel{3!}}{8\cdot 7\cdot \cancel{6}}$$

$$= \frac{1}{8\cdot 7} = \frac{1}{56}$$

Ex 2 Revisited: The top 3 students are a randomly chosen set of 3 out of 8.

We asked: What is the probability that a specific set is chosen?

How many choices of 3 out of 8 are possible?

1.) Choose 3 out of 8 in sequence: How many poss?

$$8 \cdot 7 \cdot 6$$

2.) A set of 3 can be obtained in $3!$ ways

a, b, c b, c, a
a, c, b c, a, b
b, a, c c, b, a

In total:
$$\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{(8 \cdot 7 \cdot 6) \cdot (5 \cdot 4 \cdot \ldots \cdot 1)}{(3 \cdot 2 \cdot 1) \cdot (5 \cdot 4 \cdot \ldots \cdot 1)}$$
$$= \frac{8!}{3!\,5!} = \binom{8}{3} = \binom{8}{5}$$

Generalize: Choose r out of n

$$\frac{n!}{r!\,(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$

"n choose r"

"binomial coefficients"

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots$$

We reach a node in the G. board by making a number of choices of going L or R, which correspond to choices of x and y

Pascal's triangle:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

only one path leading to a pin on the sides

each node has the # of paths leading to it in the G. board

Example 9: Given 5 men, 8 women. Randomly select 5 persons.

$$P(\text{"2 men, 3 women are select"}) =$$

S = all possible choices of 5 out of 13

$$\# S = \binom{13}{5}$$

E = selection 2 men out of 5,
3 women out of 8

$$\# E = \binom{5}{2} \cdot \binom{8}{3}$$

$$P(E) = \frac{\binom{5}{2} \cdot \binom{8}{3}}{\binom{13}{5}} \underset{\text{on handw. notes}}{=} \frac{560}{1287}$$

Example 10: Given objects $1, \dots, n$. Select subset of size k.

$$P(\underbrace{\text{"1 is in the selection"}}_{E}) =$$

S = all subsets of size k of $\{1, \dots, n\}$

$$\# S = \binom{n}{k}$$

E = all subsets of size k <u>containing 1</u>

$$\# E = \binom{n-1}{k-1}$$

i.e., only choose remaining $k-1$ out of $n-1$

$$P(E) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\cancel{(n-1)!}\; k\cancel{!}\; \cancel{(n-k)!}}{\cancel{(k-1)!}\,\underbrace{\cancel{(n-1-(k-1))!}}_{\cancel{(n-k)!}}\; n\cancel{!}} = \frac{k}{n}$$

1.4 Conditional Probabilities

Throw two dice

$$P(D_1 + D_2 = 8) = \frac{\#\{(2,6),(3,5),(4,4),(5,3),(6,2)\}}{\#\mathcal{S}}$$

$$= \frac{5}{36}$$

(What if $D_1 = 1$? $\Rightarrow P(\cdot) = 0$)

Suppose, we know $D_1 = 3$.

$$P(D_1 + D_2 = 8 \mid \underbrace{D_1 = 3}_{\text{condition}}) = ?$$

New sample space $\mathcal{S}' = \{(3,1),(3,2),\dots,(3,6)\}$

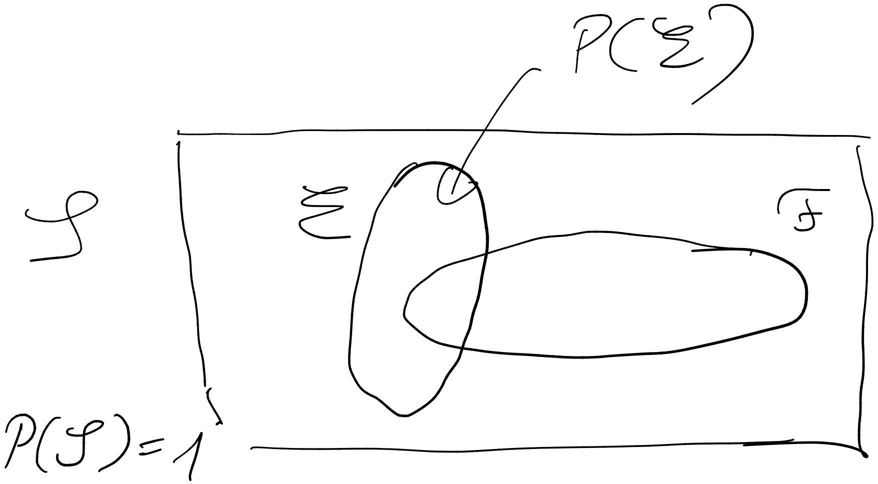
$$\#\mathcal{S}' = 6$$

$$\mathcal{E}' = \{(3,5)\} \Rightarrow \#\mathcal{E}' = 1$$

$$P(D_1 + D_2 = 8 \mid D_1 = 3) = \frac{\#\mathcal{E}'}{\#\mathcal{S}'} = \frac{1}{6}$$

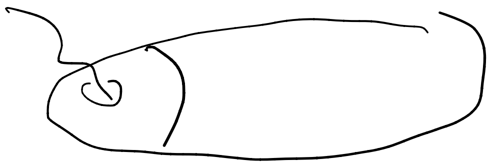
Definition 11: Events $\mathcal{E}, \mathcal{F}$, $P(\mathcal{F}) > 0$

$$P(\mathcal{E} \mid \mathcal{F}) = \frac{P(\mathcal{E} \cap \mathcal{F})}{P(\mathcal{F})}$$



Now, only outcomes in $\mathcal{F}$ are considered possible

$\mathcal{E}' = \mathcal{E} \cap \mathcal{F}$



$\mathcal{S}' = \mathcal{F}$

Normalize P to $P' = P(\cdot \mid \mathcal{F})$, such that $P(\mathcal{S}') = 1$

$$P'(\mathcal{E}) = P(\mathcal{E} \mid \mathcal{F}) = \frac{P(\mathcal{E} \cap \mathcal{F})}{P(\mathcal{F})}$$