Example 5 Box with 5 black socks, 6 white

Randomly pick two in sequence.

$$P(\underbrace{\text{2 socks have same colour}}_{\mathcal{E}}) = \frac{\#\mathcal{E}}{\#\mathcal{S}} = \frac{6}{11}$$

$\#\mathcal{E}$: 1st black, 2nd white: $5 \cdot 6$

1st white, 2nd black: $6 \cdot 5$

$\#\mathcal{S}$: 11 socks, pick first, then second: $11 \cdot 10$

$$\frac{\#\mathcal{E}}{\#\mathcal{S}} = \frac{5 \cdot 6 + 6 \cdot 5}{11 \cdot 10} = \frac{60}{110} = \frac{6}{11} \qquad \square$$

Problem: Arrange n objects on a line

a, b, c $(n=3)$:

a b c

a c b

b a c

b c a

c a b

c b a $3 \cdot 2 \cdot 1 = 6$ permutations

In general: $n(n-1)(n-2) \cdots 2 \cdot 1 = n!$

Example 6   10 books: 4 Cs, 3 Math, 2 Stat, 1 Hist

Organize so that books of same subject
are together, e.g.: SS CCCC H MMM
How many possibilities?

1) permutation of subjects
2) permutation of books within subject

Answer: $4!\,4!\,3!\,2!\,1! = (2^3\cdot 3)\cdot(2^3\cdot 3)\cdot(2^1\cdot 3)\cdot 2^1$

$= 2^8 \cdot 3^3 = 256 \cdot 27 \approx 250 \cdot 28$

$= 250 \cdot 4 \cdot 7 = 7000$

Example 7   Course with 5 male, 3 female students.
Exam with 8 different marks.

$P(\text{all female students ranked top}) = \ ?$

rankings: $8!$
permutations of female students $3!$
——— " ——— male ——— " — $5!$

$$P(\cdot) = \frac{3!\ 5!}{8!} = \frac{3\cdot 2\cdot 1}{8\cdot 7\cdot 6} = \frac{1}{56}$$

Example 7 Revisited: The top 3 students are a randomly chosen set of 3 out of 8. How many such choices are possible?

1.) Choose 3 out of 8 in sequence:

$8 \cdot 7 \cdot 6$ possibilities

2.) A set of 3 can be obtained in $3 \cdot 2 \cdot 1$ ways

E.g., abc, acb, ..., cba

In total $$\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{(8 \cdot 7 \cdot 6)(5 \cdots 1)}{(3 \cdot 2 \cdot 1) \cdot (5 \cdots 1)} = \frac{8!}{3!\,5!}$$

$\square$

Generalize: choose r out of n

$$\frac{n!}{r!\,(n-r)!} =: \binom{n}{r}$$ "n choose r"

Number of sets of size r that one can choose from a set of size n.

Example 9: Given 5 men, 8 women. Randomly select 5 persons

P(2 men, 3 women are selected) = ?

$\mathcal{S}$ = all selections of 5 out of 13 $\Rightarrow \# \mathcal{S} = \binom{13}{5}$

$\mathcal{E}$ = select 2 men out of 5, 3 women out of 8

$\Rightarrow \# \mathcal{E} = \binom{5}{2}\binom{8}{3}$

$$P(\cdot) = \frac{\binom{5}{2}\binom{8}{3}}{\binom{13}{5}} = \frac{(\cancel{5}\cdot\cancel{4}^{2})\cdot(8\cdot7\cdot\cancel{6})\;(5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}^{2}\cdot1)}{(\cancel{2}\cdot1)\cdot(\cancel{3}\cdot\cancel{2}\cdot1)\;(13\cdot\cancel{12}\cdot11\cdot\cancel{10}\cdot9)}$$

$$= \frac{8\cdot7\cdot5\cdot2}{13\cdot11\cdot9} = \frac{560}{13\cdot99} = \frac{560}{1300-13}$$

$$= \frac{560}{1287} \qquad \square$$

<u>Example 10:</u> Given n objects $1,\dots,n$. Select k.

$P($ 1 is in the selection $) = ?$

$\mathcal{S}$ = selections of k out of n $\Rightarrow \#\mathcal{S} = \binom{n}{k}$

$\mathcal{E}$ = selection of k, including $1 \Rightarrow \#\mathcal{E} = \binom{n-1}{k-1}$

$$P(\cdot) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{(n-1)!\quad (n-k)!\;k!}{(k-1)!\,(n-1-(k-1))!\quad n!} = \frac{k}{n} \qquad \square$$

1.4 Conditional Probability

Throw two dice:

$$P(D_1 + D_2 = 8) = \frac{\#\{(2,6),(3,5),(4,4),(5,3),(6,2)\}}{\# S}$$

$$= \frac{5}{36}$$

Suppose, we know that $D_1 = 3$.

$$P(D_1 + D_2 = 8 \mid D_1 = 3) = ?$$

condition

New sample space $S' = \{(3,1), \dots, (3,6)\}^*$, $\# S' = 6$

New event $E' = \{(3,5)\} \Rightarrow \# E' = 1$

$$P(D_1 + D_2 = 8 \mid D_1 = 3) = \frac{1}{6}$$

<u>Definition 11</u>: Events E, F, $P(F) > 0$.

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

is the <u>conditional probability</u> of E given F

Subjective view: update beliefs

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* Note $S' = $ "$D_1 = 3$"