1. Introduction to Probability Theory

P. T. provides
- concepts to speak about uncertainty
  (games of chance, recurring events with some pattern of variation, e.g., measurements, balls in a Galton board, etc.)
- methods to quantify uncertainty

Example: Rolling a die D

What is $P(D=6)$? $\frac{1}{6}$ $P(D \geq 4)$? $\frac{3}{6} = \frac{1}{2}$ $P(D<4)$? $\frac{1}{2}$

Meaning:

1) In the long run, 1/6 of the throws results in 6.
2) There is a chance of 1 in 6 that this throw results in 6.

frequentist subjective (Bayesian)

view

No consequences for mathematical theory

1.1 Events

Experiments:

S <u>sample space</u> (= set of possible outcomes)

S is known, but not a specific outcome

Rolling a die : $S = \{1, 2, 3, \dots, 6\}$ $\qquad \# S = 6$

Rolling two dice: $S = \{ (1,1), (1,2), \dots, (1,6),$ $\qquad \# S = 6 \cdot 6 = 36$

$\dots \qquad \dots (3,4) \dots$

$(6,1), \dots \dots (6,6) \}$

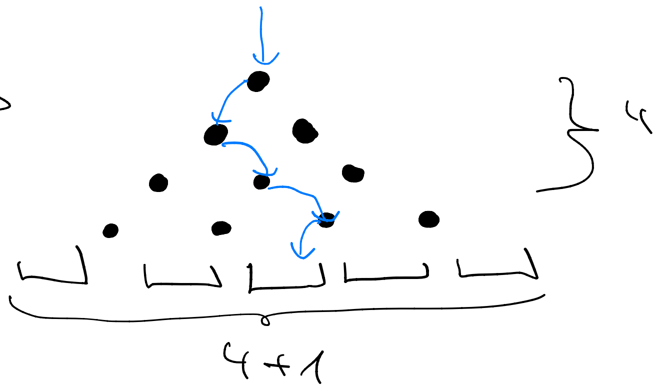
Galton board:

with n levels

has outcomes

$\{1, \dots, n+1\}$

$\{0, \dots, n\}$



S can also be infinite!

- measuring distances: any number $m > 0$ is poss. outcome
  $\quad$ ____ ; ____ $m \in [0.5, 3.00]$

  infinitely many reals in the interval

- How many times do we have to throw a die until a 6 appears
  $\Rightarrow$ consider arbitrarily long sequences of throws, even infinite one

Events:

throw an even number $\mathcal{E}_{even} = \{2, 4, 6\}$

throw two equal numbers $\mathcal{E}_{equal} = \{(1,1), (2,2), \dots, (6,6)\}$

In general: $\mathcal{E} \subseteq \mathcal{S}$ events are sets

$\mathcal{E}$ occurs if outcome $\in \mathcal{E}$ the outcome of our experiment

in an experiment

Set operations on events

$\mathcal{E} \cup \mathcal{F}$ union ("or")

$\mathcal{E} \cap \mathcal{F}$, $\mathcal{E}\mathcal{F}$ intersection ("and")

$\overline{\mathcal{E}}$ complement ("not"), $\mathcal{S} \setminus \mathcal{E} = \overline{\mathcal{E}}$

$\mathcal{E} \setminus \mathcal{F}$ $\mathcal{E}$ and not $\mathcal{F}$

$= \mathcal{E} \cap \overline{\mathcal{F}}$

Example: $\mathcal{E}_{prime} := \{2, 3, 5\}$

$\mathcal{E}_{even} \cup \mathcal{E}_{prime} = \{2, 3, 4, 5, 6\} = \overline{\{1\}}$

$\mathcal{E}_{even} \cap \mathcal{E}_{prime} = \{2\}$

$\overline{\mathcal{E}}_{prime} = \{1, 4, 6\}$

Special case $\emptyset \subseteq \mathcal{S}$, impossible event

Terminology

$\mathcal{E} \cap \mathcal{F} = \emptyset$ disjoint

$\overline{\mathcal{S}} = \emptyset$, $\overline{\emptyset} = \mathcal{S}$

$\overline{\overline{\mathcal{E}}} = \mathcal{E}$

Pronunciation

$\overline{\mathcal{E} \cup \mathcal{F}} = \overline{\mathcal{E}} \cap \overline{\mathcal{F}}$, $\overline{\mathcal{E} \cap \mathcal{F}} = \overline{\mathcal{E}} \cup \overline{\mathcal{F}}$ (De Morgan's Rule)

$\mathcal{E} \equiv \mathcal{F} \iff \mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{E}$

$\mathcal{E}, \mathcal{F}$ are <u>equivalent</u>

1.2 Axioms of Probability

$P(\mathcal{E})$ probability of $\mathcal{E}$, is a real number

A1: $0 \leq P(\mathcal{E}) \leq 1$

A2: $P(\mathcal{S}) = 1$

pairwise

any two are disjoint

A3: If $\mathcal{E}_1, \mathcal{E}_2, \ldots$ are mutually disjoint (i.e., $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $i \neq j$)

then $P(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n) = P(\mathcal{E}_1) + \cdots + P(\mathcal{E}_n) = \sum_{i=1}^{n} \mathcal{E}_i$, finite

(also holds for infinite sums)

What is $P(\emptyset) = 0$ because

$\mathcal{S}, \emptyset$ are disjoint

$$P(\mathcal{S}) = P(\mathcal{S} \cup \emptyset) = P(\mathcal{S}) + P(\emptyset)$$

$$\Rightarrow 0 = P(\emptyset)$$

$$P(\mathcal{E}_{even} \cup \{1\}) = P(\mathcal{E}_{even}) + P(\{1\})$$

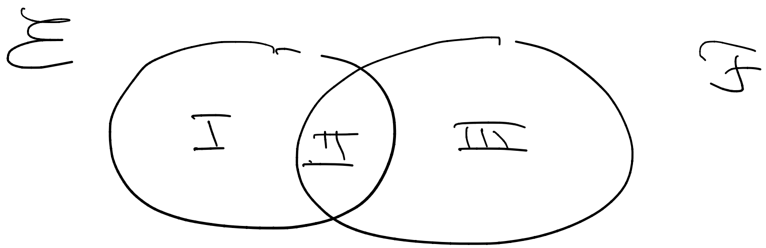
Proposition 1: $P(\bar{E}) = 1 - P(E)$

Proof: $E \cap \bar{E} = \emptyset$, $E \cup \bar{E} = S$

$$1 \underset{Ax2}{=} P(S) = P(E \cup \bar{E}) \underset{Ax3}{=} P(E) + P(\bar{E})$$

$$\Rightarrow \quad 1 - P(E) = P(\bar{E})$$

What about $P(E \cup F)$ in general?



$\mathrm{I} = E \setminus F$

$\mathrm{II} = E \cap F$

$\mathrm{III} = F \setminus E$

$$P(E \cup F) = P(\mathrm{I} \cup \mathrm{II} \cup \mathrm{III})$$
$$= P(\mathrm{I}) \cup P(\mathrm{II}) \cup P(\mathrm{III}) \quad (Ax\ 3)$$

$$P(E) = P(\mathrm{I}) + P(\mathrm{II})$$

$$P(F) = P(\mathrm{II}) + P(\mathrm{III})$$

$$\Rightarrow P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Proposition 2 : $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Example 3 :

Drinkers Quiz

B drinks beer, W drinks wine

$$P(B \cup W) = P(B) + P(W) - P(BW)$$

$$= .68 + .49 - .35 = 1.17 - .035 = .82$$

$$P(\overline{B \cup W}) = 1 - .82 = .18$$

Definition 4: The odds of $\mathcal{E}$ is

$$\frac{P(\mathcal{E})}{P(\overline{\mathcal{E}})} = \frac{P(\mathcal{E})}{1 - P(\mathcal{E})}$$

Says how much more likely $\mathcal{E}$ is than $\overline{\mathcal{E}}$

Quiz: Odds of throwing a 4

$$\mathcal{E} = \{4\}, \quad P(\mathcal{E}) = \frac{1}{6}$$

$$\text{odds}(\mathcal{E}) = \frac{P(\mathcal{E})}{1 - P(\mathcal{E})} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$$

Trump example: $P(\underbrace{\text{"T will be el."}}_{\mathcal{E}}) = .4$

$$\text{odds}(\mathcal{E}) = \frac{.4}{.6} = \frac{2}{3}$$

1.3 Uniformity

Often: all outcomes are equally likely (only possible if S finite)

and have prob. > 0

$\#S = n$, say $S = \{1, \dots n\}$

$\Rightarrow P(\{1\}) = P(\{2\}) = \dots = P(\{n\}) = p$

$\Rightarrow \underset{A2}{1 = P(S)} = P(\{1, \dots, n\}) \underset{A3}{=} P(\{1\}) + \dots + P(\{n\}) = n \cdot p$

$\Rightarrow 1 = n \cdot p \Rightarrow p = \frac{1}{n} = P(\{i\})$, $1 \le i \le n$

Generalize: $\boxed{E \subseteq S} \Rightarrow P(E) = \boxed{\frac{\#E}{n}}$ $\#E$ is the card. of E

Counting Principle

\# outcomes : throw 2 or 3 dice

first throw a die, determines a stack of cards

then pick a card (of 32)

Combinations of experiments, sequential executions

E_1 has m outcomes, $E2$ has n outcomes

\# outcomes of "$E1$ then $E2$" $= m \cdot n$

Outcomes correspond to matrix

$$\begin{matrix} (1,1), & \dots & (1,n) \\ (2,1) & & \\ \vdots & & \\ (m,1) & \dots & (m,n) \end{matrix}$$

In general: E_i has n_i outcomes

$\Rightarrow$ E_1 then E_2 ... then E_r has $n_1 \cdot \dots \cdot n_r$ outcomes