Lecture Notes

Tue, 18/01/22

- Hypothesis Testing and p-values

- t-Distribution

Wrap Up:

- Hypothesis Testing
- t-Distribution

New method has an effect on a quantity (e.g., % of people infected).

The quantity is seen as a RV. X

New method: X has a mean μ

Old method: X_0 has a mean μ_0

H_0: 4 nothing has changed: $\mu = \mu_0$ (Null hypothesis, devil's advocate appr.)

H_a: Alternative h: $\mu \neq \mu_0$

We reject H_0 if our observations are very unlikely under the cond. H_0.

Significance level

"Very unlikely" expressed as $0 < \alpha < 1$, often $\alpha = 1\%$, $\alpha = 5\%$,

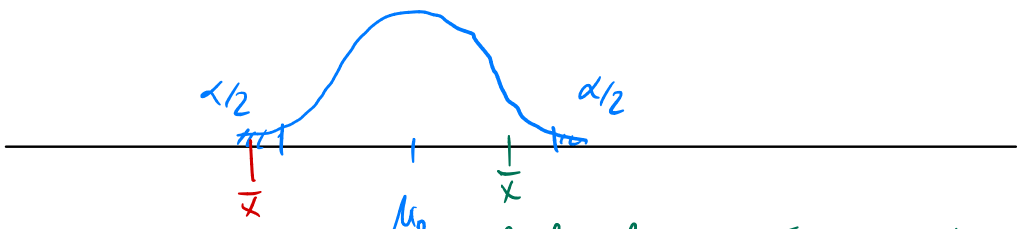
Reject H_0 if obs. has prob $< \alpha$ under the cond of H_0

Observations: average of measurements, $\bar{X}_n$

let's suppose we know stand. dev. of $\bar{X}_n$.

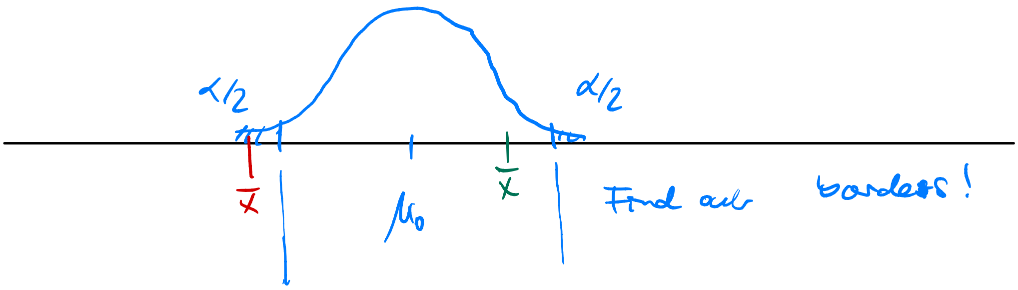
Assume $\bar{X}_n$ is norm. distributed

n measurements give us number $\bar{x}$



Prob. of $\bar{x}$ is so low, that we drop H_0

Prob. of seeing $\bar{x}$ is $> \alpha$ $\Rightarrow$ Keep H_0

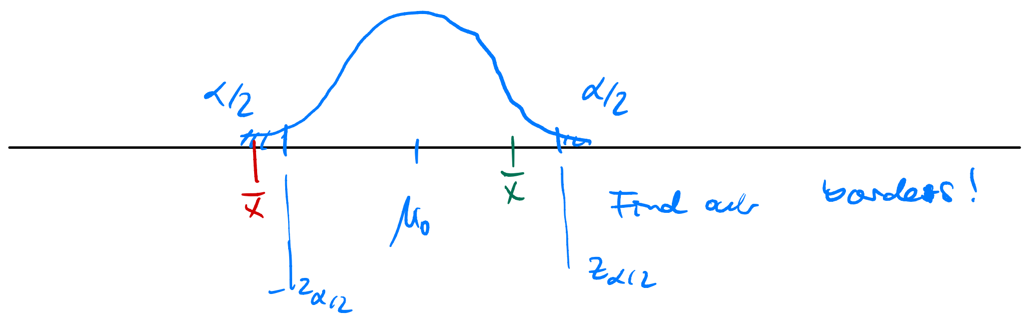


Find our borders!

If X_i have mean μ_0 $\Rightarrow$ $\bar{X}_n$ has mean μ_0

std dev σ $\Rightarrow$ $\bar{X}_n$ has std dev $\frac{\sigma}{\sqrt{n}}$

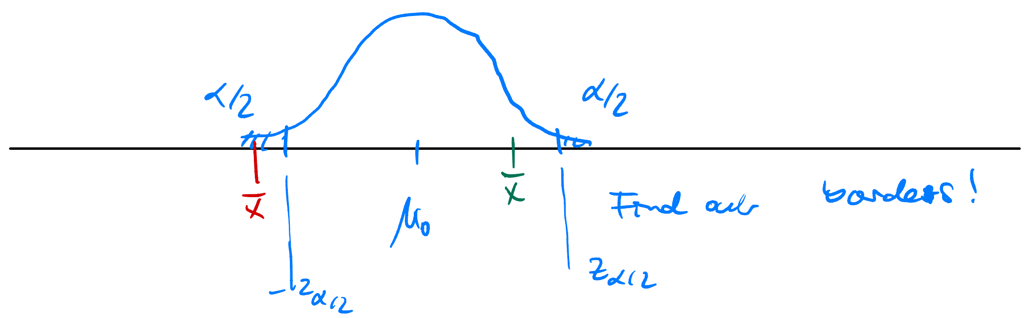
H_0: $\bar{X}_n \sim N(\mu_0, \frac{\sigma^2}{n})$, $\bar{x}$ is the avg. of meas.



If $\mu_0 = 0$, $\frac{\sigma}{\sqrt{n}} = 1$, i.e. $\bar{X}_n \sim N(0,1)$, then the borders are $-z_{\alpha/2}$, $z_{\alpha/2}$

If μ_0, $\frac{\sigma}{\sqrt{n}}$ are arbitrary, then the borders

$$\mu_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \quad \mu_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}$$



Alternative appr.: Transform the $\bar{x}$ into the setting of $N(0,1)$.

What is the prob. to see a mean as extreme as $\bar{x}$?

$$P\left[\,|\bar{X}_n - \mu_0| \geq |\bar{x} - \mu_0| \;\middle|\; \mu = \mu_0\right]$$

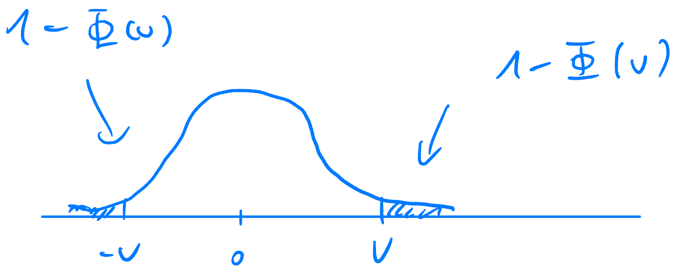
$$= P\left[\frac{|\bar{X}_n - \mu_0|}{\sigma/\sqrt{n}} \geq \frac{|\bar{x} - \mu_0|}{\sigma/\sqrt{n}} \;\middle|\; \mu = \mu_0\right]$$

$$= P\left[\,|Z| \geq \sqrt{n}\,\frac{|\bar{x} - \mu_0|}{\sigma} \;\middle|\; \mu = \mu_0\right]$$

$$= P\left[\, |Z| \geq \underbrace{\sqrt{n}\,\frac{|\bar{x}-\mu|}{\sigma}}_{v \text{ test statistic}} \;\middle|\; \mu = \mu_0 \right]$$

$$= P[\,|Z| \geq v\,] = 2(1-\Phi(v))$$

If $\overbrace{P[\,|Z| \geq v\,]}^{p_v} \leq \alpha,$ then we reject H_0, otherwise we keep it



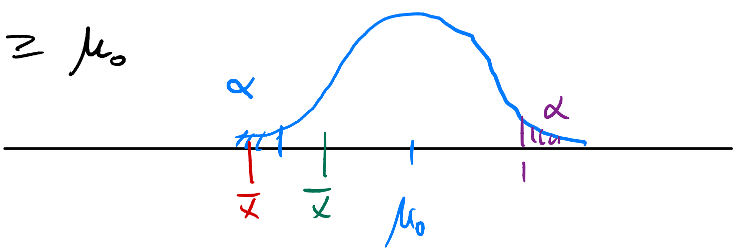
p_v is the p-value of our exp.

If $p_v \leq \alpha$, then reject H_0 otherwise accept H_0

One-sided Test:

We hope that $\mu < \mu_0$. We play devil's advocate:

H_0: $\mu \geq \mu_0$



$$P[\,\bar{X}_n \leq \bar{x} \mid \mu = \mu_0\,]$$

$$= P\left[\sqrt{n}\,\frac{\bar{X}_n-\mu}{\sigma} \leq \sqrt{n}\,\frac{\bar{x}-\mu}{\sigma} \;\middle|\; \mu = \mu_0\right]$$

$$= P\left[Z \leq \sqrt{n}\,\frac{\bar{x}-\mu}{\sigma} \;\middle|\; \mu = \mu_0\right]$$

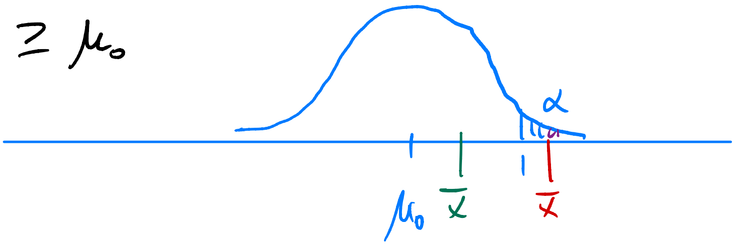
$$= P\left[Z \leq \underbrace{\sqrt{n}\,\frac{\bar{x}-\mu_0}{\sigma}}_{v \text{ test statistic}}\right] = P[Z \leq v] = \Phi(v) = 1-\Phi(-v)$$

p-value of one-sided left-sided test

One-sided Test:

We hope that $\mu < \mu_0$. We play devil's advocate:

H_0: $\mu \geq \mu_0$



$$P[\bar{X}_n \geq \bar{x} \mid \mu = \mu_0]$$

$$= P\left[\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \geq \sqrt{n}\frac{\bar{x} - \mu}{\sigma} \,\middle|\, \mu = \mu_0\right]$$

p-value of one-sided right-sided test

$$= P\left[Z \geq \sqrt{n}\frac{\bar{x} - \mu}{\sigma} \,\middle|\, \mu = \mu_0\right]$$

$$= P\Big[Z \geq \underbrace{\sqrt{n}\frac{\bar{x} - \mu_0}{\sigma}}_{v \text{ test statistic}}\Big] = P[Z \geq v] = 1 - \Phi(v)$$

$$S = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\sqrt{n}\frac{\bar{x} - \mu_0}{\sigma} \longrightarrow \sqrt{n}\frac{\bar{x} - \mu_0}{S}$$ observation of

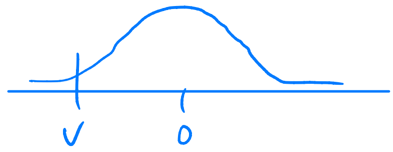
$$\sqrt{n}\frac{\bar{X}_n - \mu_0}{S} \sim t_{n-1}$$

if $X_i \sim N(\mu_0, \sigma^2)$

Consequence: When it comes to computing prob.s, then replace $N(0,1)$ with t_{n-1}

Weight of Chicken

$\mu_0 = 3.8$ $\qquad$ $H_0: \mu \geq \mu_0$



$\sigma = 0.6$

$n = 16, \quad \bar{x} = 3.55$

let's compute the p-value for this experiment

Test statistic

$$V = \sqrt{n}\,\frac{\bar{x} - \mu_0}{\sigma} = \sqrt{16}\,\frac{3.55 - 3.8}{0.6} = 4\,\frac{-0.25}{0.6}$$

$$= -\frac{1}{0.6} = -\frac{10}{6} \approx -1.\bar{6} = -1.667$$

p-value : $P[Z \leq v] = 1 - \Phi(-v) = 1 - \Phi(1.667)$

$$= 1 - 0.9525 = 0.0475 = 4.75\,\%$$

$p_v = 4.75\,\%$

We know σ!

$\alpha = 5\% \Rightarrow$ reject H_0

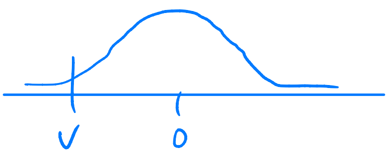
$\alpha = 1\% \Rightarrow$ accept H_0

If we don't know σ and $s = 0.6$ kg :

$v = -1.667$ is still the same,

but now we t-distribution

From the t-table we see that



- $0.05 < p_v < 0.1$
- $p_v = 0.058$

We accept H_0 for both significance levels.