Data Structures and Algorithms

Chapter 7

Graphs

Werner Nutt

Acknowledgments

- The course follows the book “Introduction to Algorithms‘”, by **Cormen, Leiserson, Rivest and Stein**, MIT Press [CLRST]. Many examples displayed in these slides are taken from their book.
- These slides are based on those developed by Michael Böhlen for this course.

(See http://www.inf.unibz.it/dis/teaching/DSA/)

- The slides also include a number of additions made by Roberto Sebastiani and Kurt Ranalter when they taught later editions of this course

(See http://disi.unitn.it/~rseba/DIDATTICA/dsa2011_BZ//)

DSA, Chapter 7: Overview

1. Graphs – Principles
2. Graph representations
3. Traversing Graphs
   - Breadth-First Search
   - Depth-First Search
4. DAGs and Topological Sorting

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Graphs – Definition

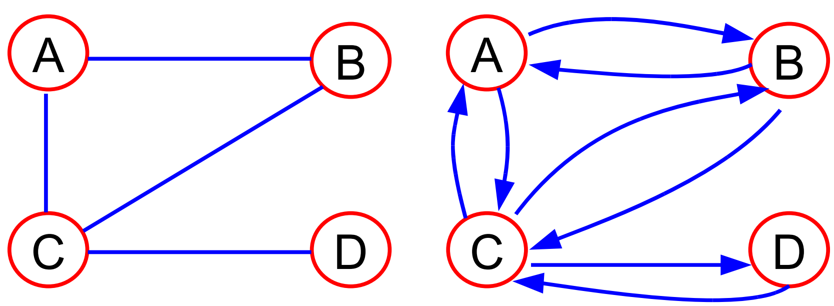
A graph $G = (V,E)$ is composed of

- a set of vertices V
- a set of *edges* $E \subset V \times V$ connecting the vertices

An edge $e = (u,v)$ is a pair of vertices

We assume directed graphs

- if a graph is undirected,
  we represent an edge between u and v
  by two pairs $(u,v) \in E$ and $(v,u) \in E$



V = {A, B, C, D}

E = {(A,B), (B,A), (A,C), (C,A),
(C,D), (D,C), (B,C), (C,B)}

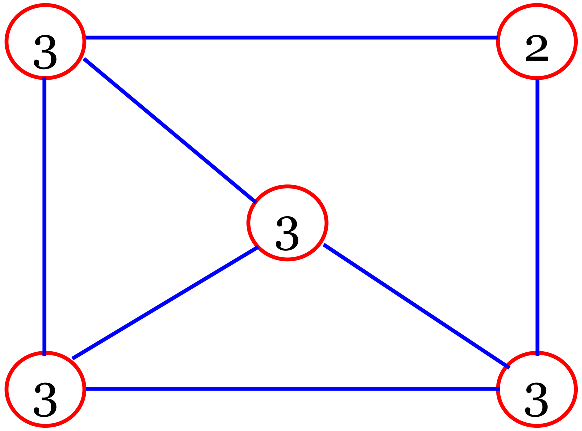
Applications

- Electronic circuits, pipeline networks
- Transportation and communication networks
- Modeling any sort of relationships
  (between components, people, processes, concepts)

Graph Terminology

A vertex v is adjacent to vertex u iff $(u,v) \in E$

The degree of a vertex: # of adjacent vertices

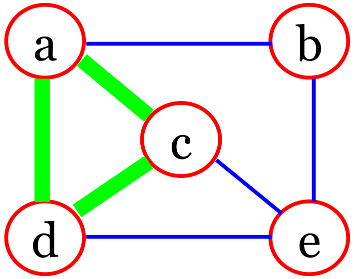


A path is a sequence of vertices $v_1, v_2, \ldots v_k$ such that v_{i+1} is adjacent to v_i for $i = 1 \ldots k - 1$

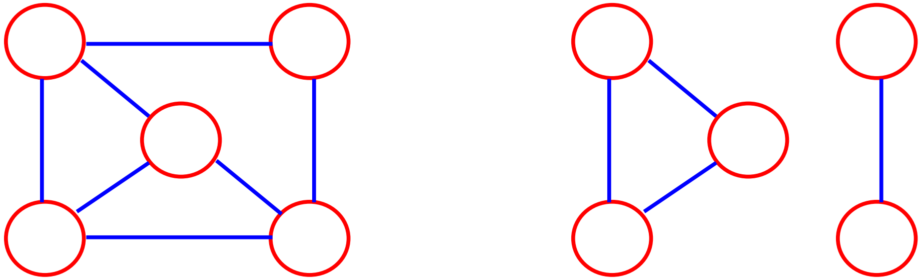
Graph Terminology/2

Simple path – a path with no repeated vertices

Cycle – a simple path, except that the last vertex is the same as the first vertex



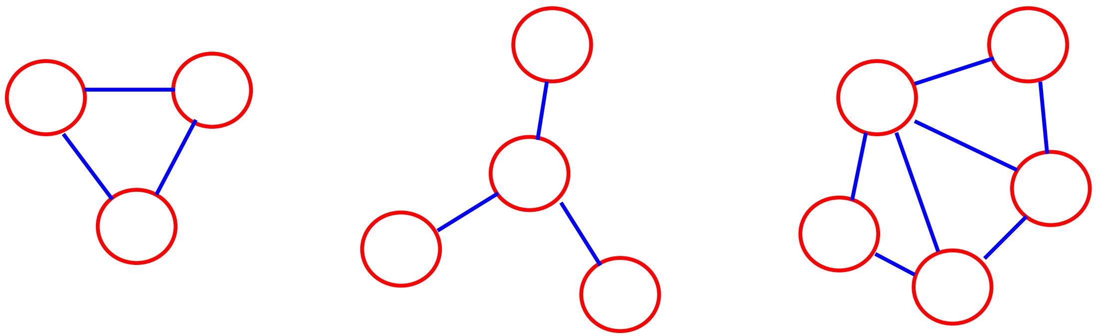
Connected graph – any two vertices are connected by some path



Graph Terminology/3

Subgraph – a subset of vertices and edges forming a graph

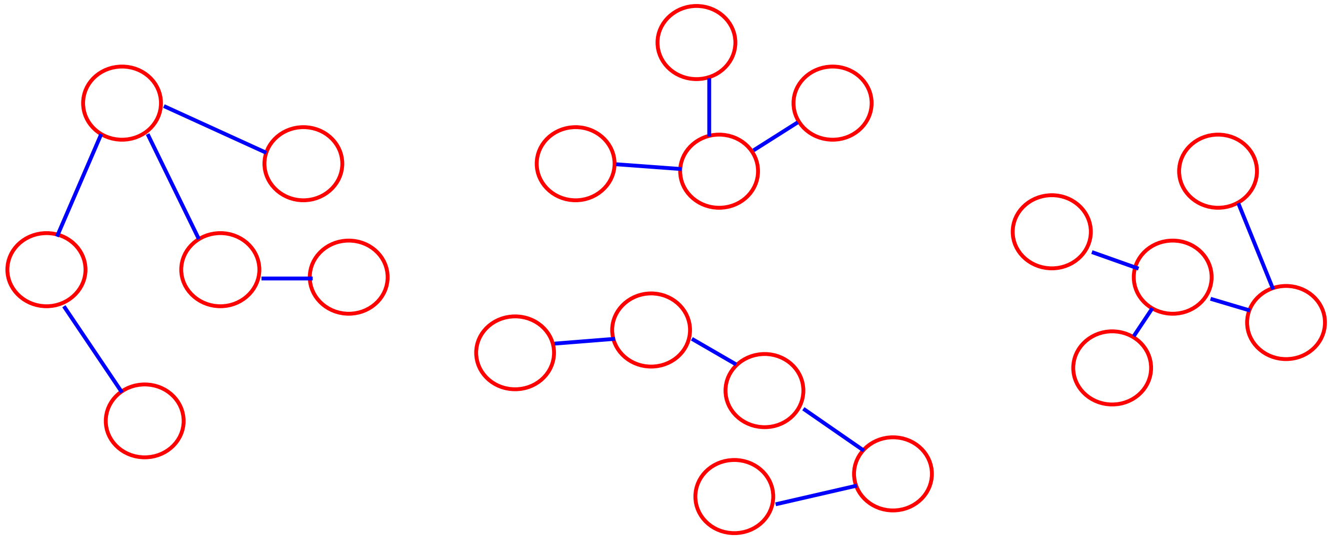
Connected component – maximal connected subgraph.
Example: the graph below has 3 connected components



Graph Terminology/4

Tree – connected graph without cycles

Forest – collection of trees



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Adjacency Matrix

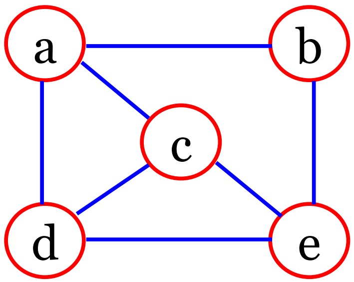
Matrix M with entries for all pairs of vertices

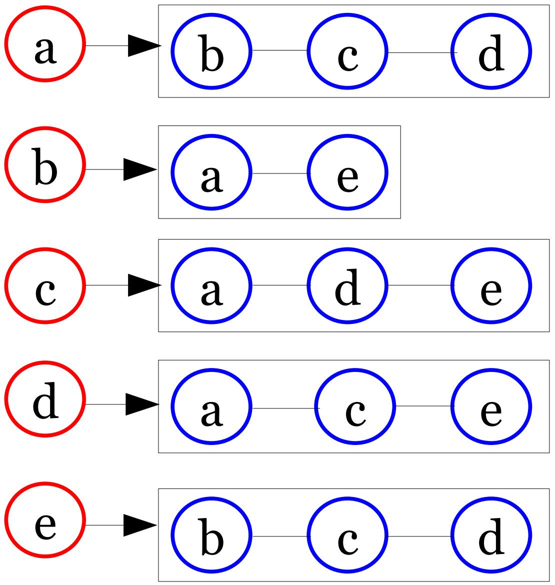
- M[i,j] = true – there is an edge (i,j) in the graph
- M[i,j] = false – there is no edge (i,j) in the graph

Needs space $\Theta(|V|^2)$

Data Structures for Graphs

- The adjacency list of a vertex *v*:
  sequence of vertices adjacent to *v*
- A graph is represented by the adjacency lists of all its vertices





- Needs space $\Theta(|V|+|E|)$

Pseudocode Assumptions

Each node has some properties (fields of a record):

- adj: list of adjacent nodes
- dist: distance from start node in a traversal
- pred: predecessor in a traversal
- color: color of the node (is changed during traversal: white, gray, black)
- starttime: time when first visited during a traversal (depth first search)
- endtime: time when last visited during a traversal (depth first search)

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Graph Searching Algorithms

- Systematic search of every edge and vertex of the graph
- Graph $G = (V,E)$ is either directed or undirected
- Applications
  - Memory management (Cheney algorithm for garbage collection)
  - Graphics (ray tracing)
  - Maze-solving
  - Networks: routing, searching, clustering, etc.

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Breadth-First Search

- A Breadth-First Search (BFS) traverses a connected component of an (un)directed graph, and in doing so defines a spanning tree.
- BFS in an undirected graph G is like wandering in a labyrinth with a string and exploring the neighborhood first.
- The starting vertex *s*, it is assigned distance 0.
- In the first round the string is unrolled 1 unit. All edges that are 1 edge away from the anchor are visited (discovered) and assigned distance 1.

Breadth-First Search/2

- In the second round, all the new edges that can be reached by unrolling the string 2 edges are visited and assigned a distance of 2
- This continues until every vertex has been assigned a level
- The label of any vertex *v* corresponds to the length of the shortest path (in terms of edges) from *s* to *v*

BFS Algorithm

```
BFS(G,s)
for u ∈ G.V do
    u.color := white
    u.dist := ∞
    u.pred := NULL
s.color := gray
s.dist := 0
Q := new Queue()     // FIFO queue
Q.enqueue(s)
while not Q.isEmpty() do
    u := Q.dequeue()
    for v ∈ u.adj do
        if v.color = white
          then v.color := gray
               v.dist := u.dist + 1
               v.pred := u
               Q.enqueue(v)
    u.color := black
```

Initialize all vertices

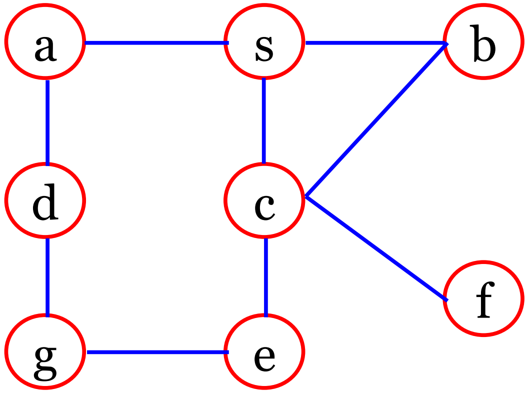
Initialize BFS with *s*

Handle all of *u*'s children before handling children of children

Coloring of Vertices

- A vertex is white if it is undiscovered
- A vertex is gray if it has been discovered but not all of its edges have been explored
- A vertex is black after all of its adjacent vertices have been discovered (the adj. list was examined completely)

Let's do an example of BFS:



BFS Running Time

Given a graph $G = (V,E)$

- Vertices are enqueued if their color is white
- Assuming that en- and dequeuing takes O(1) time the total cost of this operation is O(V)
- Adjacency list of a vertex is scanned when the vertex is dequeued
- The sum of the lengths of all lists is $\Theta(E)$ Thus, $\Theta(E)$ time is spent on scanning them
- Initializing the algorithm takes $\Theta(V)$

Total running time is $\Theta(V+E)$
(linear in the size of the adjacency list representation of G)

BFS Properties

Given a graph G = (*V*,*E*).

Then BFS

- discovers all vertices reachable from a source vertex *s,*
- computes the shortest distance to all reachable vertices (proof in textbook [CLRS]),
- computes a breadth-first tree that contains all such reachable vertices (tree property holds because pred is unique).

For any vertex *v* reachable from *s*,
the path in the breadth first tree from s to v,
corresponds to a shortest path in G

BFS Applications

- Find a shortest path,
  where length is measured in number of edges.
- Find connected components
  of an undirected graph
- Garbage collection
  (Traverse the graph of objects reachable from the stack in BFS manner.)
- Check whether a graph is bipartite
  (A graph is bipartite if its vertices can be coloured red and blue such that no edge connects vertices of the same colour.)

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Depth-First Search

A depth-first search (DFS) in an undirected graph G is like wandering in a labyrinth with a string and following one path to the end

- We start at vertex *s*, tying the end of our string to the point and painting *s* “visited (discovered)”; next we label *s* as our current vertex called *u*
- Now, we travel along an arbitrary edge (*u*,*v*)
- If edge (*u*,*v*) leads us to an already visited vertex *v*, we return to *u*
- If vertex *v* is unvisited, we unroll our string, move to *v*, paint *v* “visited”, set *v* as our current vertex, and repeat the previous steps

Depth-First Search/2

- Eventually, we will get to a point where all edges from *u* lead to visited vertices
- We then backtrack by rolling up our string until we get back to a previously visited vertex *v*
- *v* becomes our current vertex and we repeat the previous steps

DFS Algorithm

```
DFS-All(G)
for u ∈ G.V do
    u.color := white
    u.pred := NIL
time := 0
for u ∈ G.V do
    if u.color = white then DFS(u)
```

Init all vertices

Visit all vertices

```
DFS(u)
u.color := gray
time := time + 1
u.starttime := time
for v ∈ u.adj do
    if v.color = white then
        v.pred := u
        DFS(v)
u.color := black
time := time + 1
u.endtime := time
```

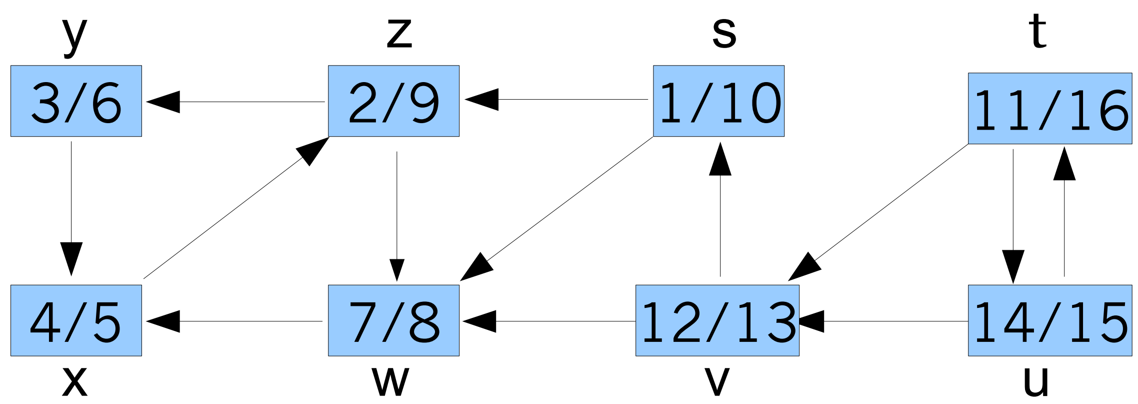
Visit all children recursively (children of children are visited first)

DFS Algorithm/2

- Initialize – color all vertices white
- Visit each and every white vertex using DFS-All (even if there are disconnected trees)
- Each call to DFS(u)
  roots a new tree of the depth-first forest
  at vertex u
- When DFS returns, each vertex *u* has assigned
  - a discovery time *d*[*u*]
  - a finishing time *f*[*u*]

Example of DFS

- Start with s:



- Explores subgraph s first, t second

DFS Algorithm: Running Time

Running time

- the loops in DFS-All take time Θ(V) each, excluding the time to execute DFS
- DFS is called once for every vertex
  - it's only invoked on white vertices, and
  - paints the vertex gray immediately
- for each DFS a loop interates over all *v*.adj

$$\sum_{v \in V} |v.adj| = \Theta(E)$$

- the total cost for DFS is Θ(*E*)
- the running time of DFS-All is Θ(*V*+*E*)

DFS versus BFS

- The BFS algorithms visits all vertices that are reachable from the start vertex.
  It returns one search tree.
- The DFS-All algorithm visits all vertices in the graph.
  It may return multiple search trees.
- The difference comes from the applications of BFS and DFS.
  This behavior of the algorithms depends on the policy according to which the next nodes to be processed are determined.

Generic Graph Search

```
GenericGraphSearch(G,s)
for each vertex u ∈ G.V do
  u.color := white; u.pred := NIL
s.color := gray
GrayVertices := new Container();
GrayVertices.addTo(s)
while not GrayVertices.isEmpty() do
  u := GrayVertices.extractFrom()
  for each v ∈ u.adj do
    if v.color = white then
      v.color := gray
      v.pred := u
      GrayVertices.addTo(v)
  u.color := black
```

- BFS if `GrayVertices` is a Queue (FIFO)
- DFS if `GrayVertices` is a Stack (LIFO)

DFS Annotations

- A DFS can be used to annotate vertices and edges with additional information.
  - starttime (when was the vertex visited first)
  - endtime (when was the vertex visited last)
  - edge classification (tree, forward, back or cross edge)
- The annotations reveal useful information about the graph that is used by advanced algorithms

DFS Timestamping

- Vertex *u* is
  - white before *u.starttime*
  - gray between *u.starttime* and *u.endtime*, and
  - black after *u.endtime*

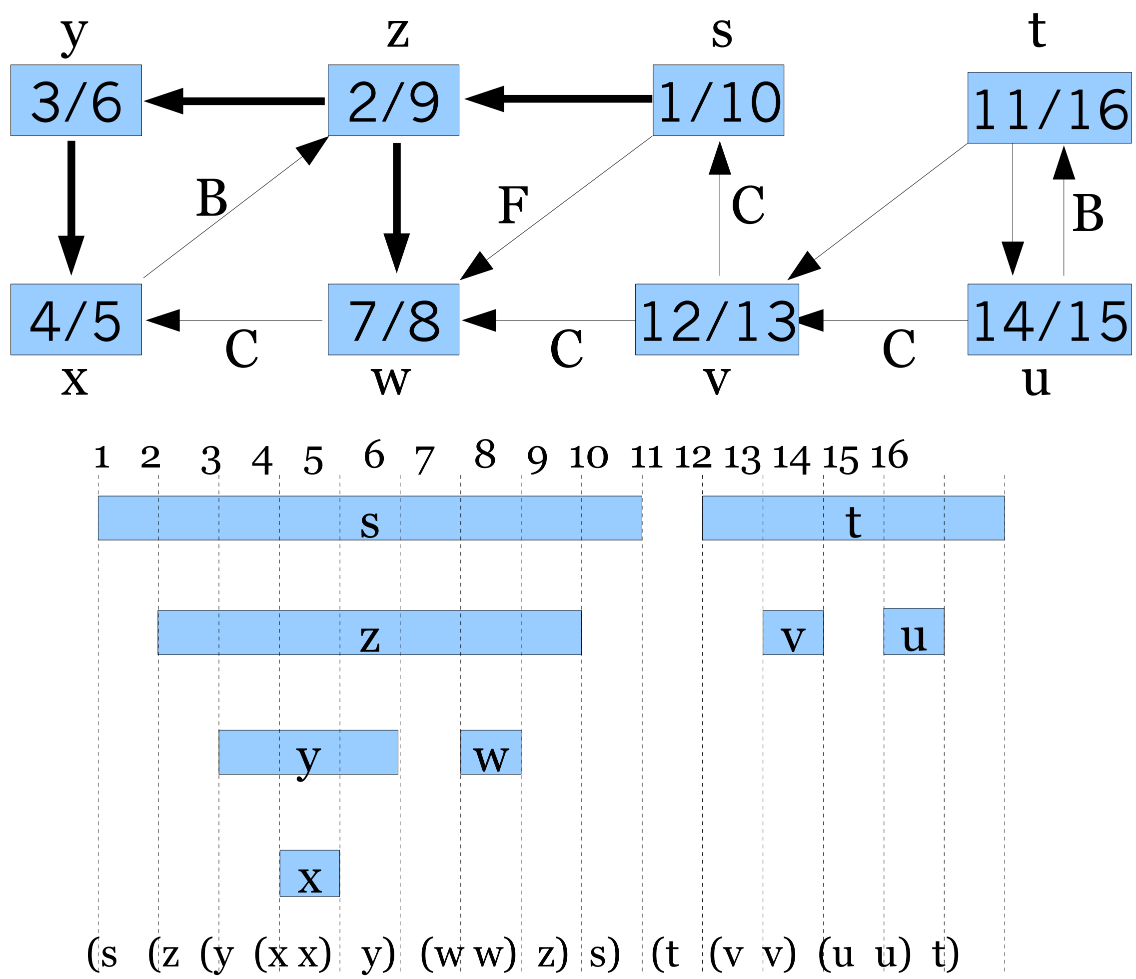
- Notice the structure throughout the algorithm
  - gray vertices form a linear chain
  - correponds to a stack of vertices that have not been exhaustively explored
    (DFS started, but not yet finished)

DFS Parenthesis Theorem

- Start and end times have parenthesis structure
  - represent starttime of *u* with left parenthesis "(u"
  - represent endtime of *u* with right parenthesis "u)"
  - history of start- and endtimes makes a well-formed expression (parentheses are properly nested)

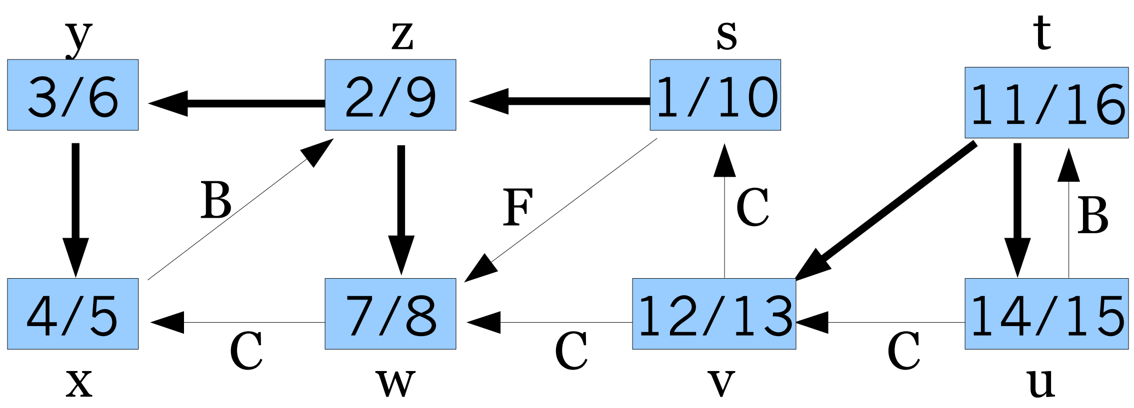
- Intuition for proof:
  any two intervals are either disjoint or enclosed
  - Overlapping intervals would mean finishing ancestor, before finishing descendant or starting descendant without starting ancestor

DFS Parenthesis Theorem/2



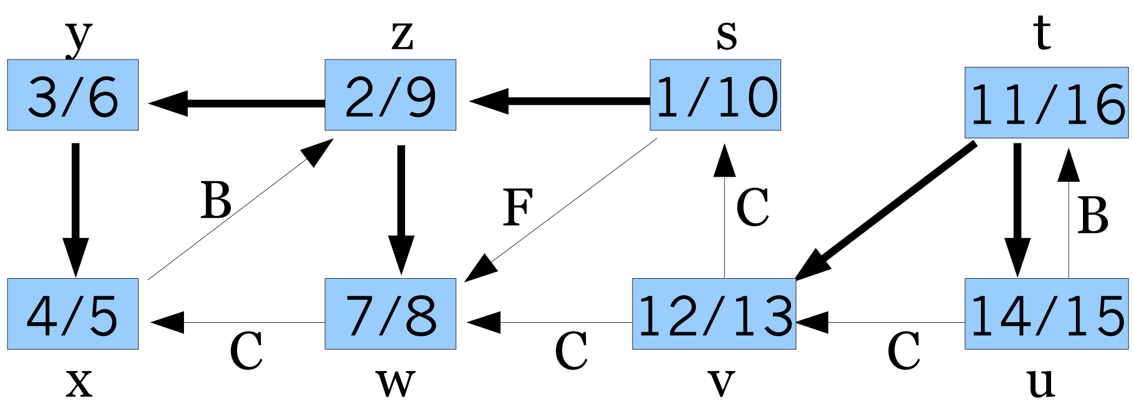
DFS Edge Classification

- Tree edge (gray to white)
  - Edges in depth-first forest
- Back edge (gray to gray)
  - from descendant to ancestor in depth-first tree
  - Self-loops



DFS Edge Classification

- Forward edge (gray to black)
  - Nontree edge from ancestor to descendant in depth-first tree
- Cross edge (gray to black)
  - remainder – between trees or subtrees



DFS Edge Classification/3

- In a DFS the color of the next vertex decides the edge type (this makes it impossible to distinguish forward and cross edges)
- Tree and back edges are important
- Most algorithms do not distinguish between forward and cross edges

DFS Applications

- Find connected components (in an undirected graph)
- Check whether a digraph is bipartite
- Check a digraph for cycles
- Find a linear order that refines a given partial order ("topological sorting" ➔ next section)

DFS is the core of other graph algorithms, e.g., for

- finding strongly connected components of a digraph
- finding bridges in a graph

Suggested exercises

- Implement BFS and DFS, both iterative and recursive
- Using paper & pencil, simulate the behaviour of BFS and DFS (and All-DFS) on some graphs, drawing the evolution of the queue/stack

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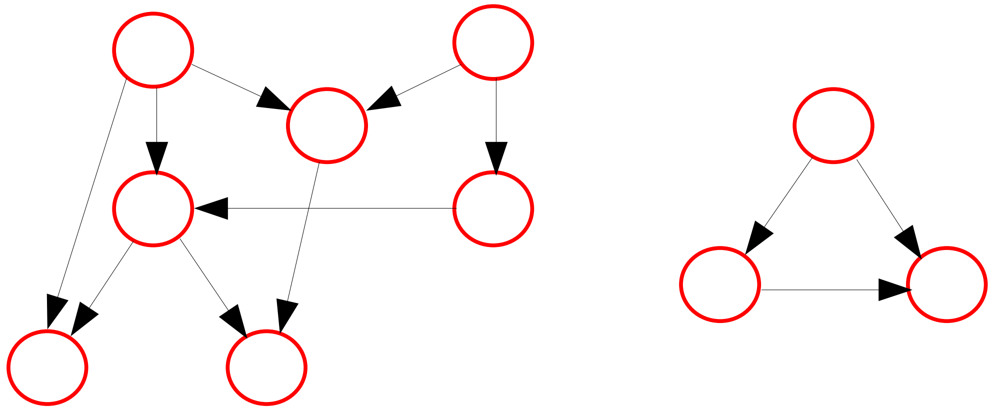
Directed Acyclic Graphs (DAGs)

- A DAG is a directed graph without cycles



- DAGs are used to indicate precedence among events (event *x* must happen before *y)*
- An example would be a parallel code execution
- We get a total order using Topological Sorting

DAG Theorem

A directed graph *G* is acyclic if and only if a DFS of *G* yields no back edges.

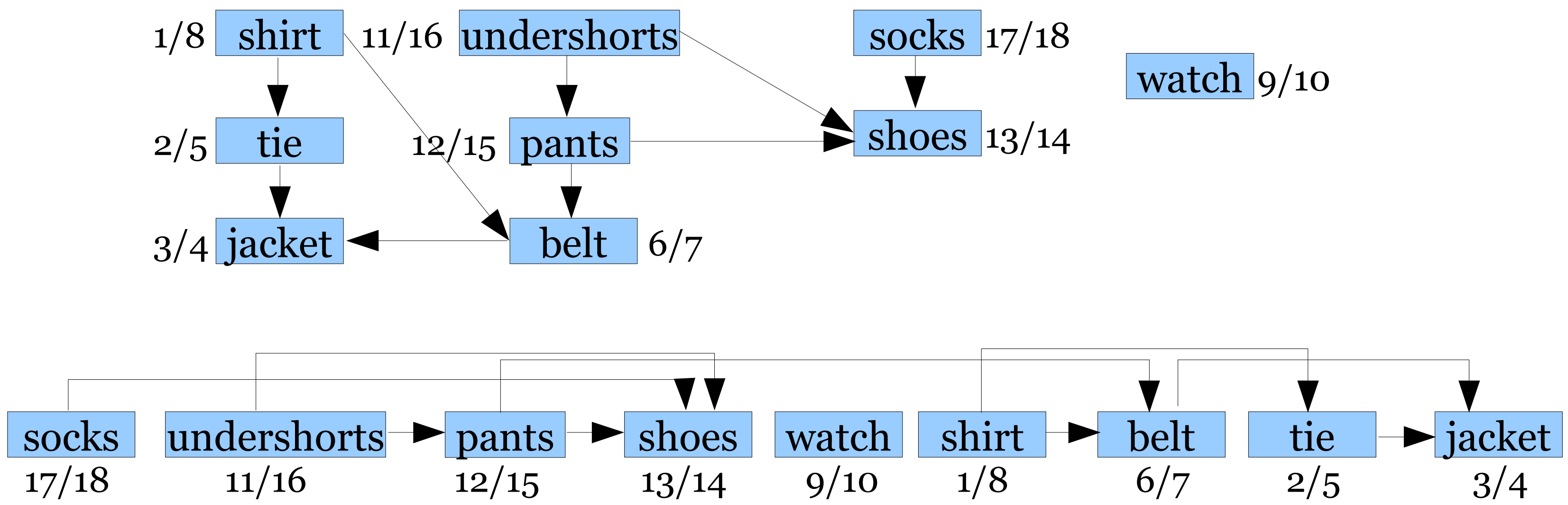
Proof: Suppose there is a back edge (*u*,*v*):
Then *v* is an ancestor of *u* in DFS forest. Thus, there is a path from *v* to *u* in G and (*u*,*v*) completes the cycle.
Suppose there is a cycle *c*: Let *v* be the first vertex in *c* to be discovered and *u* is the predecessor of *v* in *c*.

- Upon discovering *v* the whole cycle from *v* to *u* is white
- We visit all nodes reachable on this white path before DFS(*v*) returns, i.e., vertex *u* becomes a descendant of *v*
- Thus, (*u,v*) is a back edge

Thus, we can verify whether *G* is a DAG using DFS

Topological Sorting Example

- Precedence relations: an edge from *x* to *y* means one must be done with *x* before one can do *y*
- Intuition: can schedule task only when all of its precondition subtasks have been scheduled



Topological Sorting/1

- Sorting of a directed acyclic graph (DAG)
- A topological sort of a DAG
  is a linear ordering of all its vertices
  such that for any edge (u,v) in the DAG,
  u appears before v in the ordering

Topological Sorting/2

The following algorithm topologically sorts a DAG

The linked lists comprises a total ordering

```
TopologicalSort(G)
  Call DSF(G) to compute
    v.endtime for each vertex v
  As each vertex is finished,
    insert it at the beginning of a linked list
  Return the linked list of vertices
```

Remark: Of course, in practice, one need not compute the endtime, it is enough to insert each vertex when work on it is finished.

Topological Sorting Correctness

Claim: If *G* is a DAG and $(u,v) \in E$ ➔ u.endtime > v.endtime

- When (*u*,*v*) is explored, *u* is gray.
  We can distinguish three cases:
  - *v.color* = gray ➔ (*u*,*v*) is a back edge (cycle, contradiction)
  - *v.color* = white ➔ *v* becomes descendant of *u*
    ➔ *v* will be finished before *u*
    ➔ *v.endtime* < *u.endtime*
  - *v.color* = black ➔ *v* is already finished
    ➔ *v.endtime* < *u.endtime*
- The definition of topological sort is satisfied

Topological Sorting: Running Time

- Running time
  - depth-first search: $O(V+E)$ time
  - insert each of the $|V|$ vertices to the front of the linked list: $O(1)$ per insertion
- Thus the total running time is $O(V+E)$

Suggested Exercises

- Implement topological sorting,
  with a check for the DAG property

- Using paper & pencil,
  simulate the behaviour of topological sorting

Summary

- Graphs
  - $G = (V,E)$, vertex, edge, (un)directed graph, cycle, connected component, ...
- Graph representation: matrix/adjacency list
- Basic techniques to traverse/search graphs
  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)
- Topological Sorting