Probability Theory and Statistics

– Exam –

Werner Nutt

10 September 2019

Name: ______________________________

Registration Number: ______________________________

Question	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
Points	23	13	23	15	18	8	100
Reached							

Instructions

- The exam comprises 6 questions, which consist of several subquestions. You will have 2 hours time to answer the questions.
- If numerical answers are requested it suffices to write them down as arithmetic expressions, like e.g., $\frac{7 \cdot 3+5}{30}$. Expressions should be simplified as much as possible, though.
- There is a total of 100 points that can be achieved in this exam. Marking will be out of 90. That is, to achieve a final mark of 30, it will suffice to obtain 90 points.
- Please, write down the answers to your questions in the present exam booklet handed out to you.
- For drafts use the blank paper provided by the university.
- If the space in the booklet turns out to be insufficient, please use the university paper for additional answers and return them with the booklet.
- **No questions** will be answered during the exam. If you are not sure about interpreting a question, you may write down additional assumptions you made in order to proceed with your solution.

1 Drawing Balls from an Urn **(23 Points)**

Suppose we have an infinite supply of balls, colored either red or blue. The probability of drawing a red ball from the supply is equal to the probability of drawing a blue one. We draw two balls from the supply and put them into an urn. Now, we repeatedly draw a ball from the urn, note its colour, and then return to the urn.

(i) What is the probability that the first ball drawn is red?

(ii) What is the probability that the first two balls drawn are red?

Now, suppose that the first two balls drawn are in fact red.

(iii) What is the probability that both balls in the urn are coloured red?

(iv) What is the probability that the next ball drawn will be red?

(v) Generalizing, what is the probability that the next n balls drawn will be red?

Justify your answers.

Hint: Bayes' Theorem may be useful for this question.

2 Waiting Time **(13 Points)**

The waiting time (in minutes) until an event is observed is distributed as an exponential distribution with parameter $\lambda = 0.5$.

(i) On average, how much time do we have to wait to observe the event?

(ii) If the event is not observed in the first 2 minutes, how much more do you expect to wait for it?

(iii) If we run the experiment twice in parallel, find the probability that the minimum waiting time is at least 3 minutes.

Justify your answers, mentioning all assumptions made.

Hint: Remember that the density of an exponential distribution is $\lambda e^{-\lambda x}$ for $x \geq 0$.

3 Joint Distribution **(23 Points)**

The joint probability density function of $\mathcal{X}$ and $\mathcal{Y}$ is

$$f(x, y) = cxe^{-x-2y} \qquad 0 < x;\ 0 < y.$$

(i) Find the value c.

(ii) Compute the density function of $\mathcal{X}$.

(iii) Compute the density function of $\mathcal{Y}$.

(iv) Are $\mathcal{X}$ and $\mathcal{Y}$ independent? (Explain your answer.)

(v) Find $P(\mathcal{Y} < \mathcal{X})$.

Show your computations.

4 Testing Car Tyres **(15 Points)**

The manufacturer of a new car tyre claims that its average life will be at least 60,000 km.

To verify this claim a sample of 25 tyres is tested. The outcome of the test is a sample mean of 54,000 km and a sample standard deviation of 12,000 km.

(i) Find a value c such that, with probability 99%, the true mean is less than c.

(ii) Compute the p-value for the hypothesis $H_0 : \mu \geq 60,000$. Approximate it as well as you can from the probability tables provided.

(iii) What would you need to change in your calculations if we knew that the *population* standard deviation is 12,000 km? What would be the values for c in (i) and the p-value in (ii)?

Justify your answers.

5 Low and High Throws With a Fair Die (18 Points)

We throw a fair die repeatedly. If the die falls on 1, 2, or 3, we say that it is a low throw (otherwise, it is a high throw). Approximate the probability of:

(i) seeing at most 55 low throws, in 100 throws of the die,

(ii) seeing at least 215 highs, in 400 throws.

Suppose now that you win 1 Euro for each low throw, and lose 1 Euro for each high throw.

(iii) What is the probability of winning more than 10 Euros after 100 throws?

(iv) How many throws do you need to guarantee that the probability of winning at least 10 Euros is greater than 0.99?

Justify your answers.

6 Example Distribution **(8 Points)**

Provide an example distribution covered in the course (other than the exponential distribution, which occurred in the exam).

(i) Give the name and the density function of the distribution.

(ii) Briefly explain where it is applied.