Expected Values

Simple Example: 2 tosses of a coin

$$S = \left\{ \begin{array}{l} (H,H) \\ (H,T) \\ (T,H) \\ (T,T) \end{array} \right\}$$

Suppose coin is not fair:

$$P[H] = \frac{2}{3}, \quad P[T] = \frac{1}{3}$$

$$P(H,H) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$
$$P(H,T) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$
$$P(T,H) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$
$$P(T,T) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

$X : S \to \mathbb{R}$,

$X(s) := \#H$ in s

Average value of X:

$$2 \cdot \frac{4}{9} + 1 \cdot \frac{2}{9} + 1 \cdot \frac{2}{9} + 0 \cdot \frac{1}{9}$$

$$E[X] = \sum_{s \in S} X(s) \cdot P(s)$$ expected Value of X

$$S = \left\{ \begin{array}{l} (H,H) \\ (H,T) \\ (T,H) \\ (T,T) \end{array} \right\}$$

(S, P) Probability space

$X : S \to \mathbb{R}$

$$P(H,H) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$
$$P(H,T) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$
$$P(T,H) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$
$$P(T,T) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

Probability mass function of X:

$$p_X(0) = \frac{1}{9}$$
$$p_X(1) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$
$$p_X(2) = \frac{4}{9}$$

Can we recover $E[X]$ from p_X?

let $\{x_1, \dots, x_n\}$ be the values of X, which we call the range of X ($X(S)$), $Rg(X)$

$$E[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$
$$= \sum_{x \in Rg(X)} x \cdot p_X(x) = \sum_{i=1}^{n} x_i \cdot p_X(x_i)$$

let $\mathcal{Y} : S \to \mathbb{R}$, be the indicator fct for "first coin has H"

$$\mathcal{Y}(s) = \begin{cases} 1 & \text{for } (H,H), (H,T) \\ 0 & \text{for } (T,H), (T,T) \end{cases}$$

$$P(H,H) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$
$$P(H,T) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$
$$P(T,H) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$
$$P(T,T) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

$$E[\mathcal{Y}] = \sum_{s \in S} \mathcal{Y}(s) \cdot P(s) = 1 \cdot \frac{4}{9} + 1 \cdot \frac{2}{9} + 0 \cdot \frac{2}{9} + 0 \cdot \frac{1}{9}$$

let $\mathcal{W} := \mathcal{X} + \mathcal{Y}$, $\mathcal{W}(s) = \mathcal{X}(s) + \mathcal{Y}(s)$

$$E[\mathcal{W}] = \sum_{s \in S} \mathcal{W}(s) \cdot P(s) = \sum_{s \in S} (\mathcal{X}(s) + \mathcal{Y}(s)) \cdot P(s)$$
$$= (2+1) \cdot \frac{4}{9} + (1+1) \cdot \frac{2}{9} + (1+0) \cdot \frac{2}{9} + (0+0) \frac{1}{9}$$

$$E[\mathcal{Y}] = \sum_{s \in S} \mathcal{Y}(s) \cdot P(s) = 1 \cdot \frac{4}{9} + 1 \cdot \frac{2}{9} + 0 \cdot \frac{2}{9} + 0 \cdot \frac{1}{9}$$

let $\mathcal{W} := \mathcal{X} + \mathcal{Y}$, $\mathcal{W}(s) = \mathcal{X}(s) + \mathcal{Y}(s)$

$$E[\mathcal{X} + \mathcal{Y}] = E[\mathcal{W}] = \sum_{s \in S} \mathcal{W}(s) \cdot P(s) = \sum_{s \in S} (\mathcal{X}(s) + \mathcal{Y}(s)) \cdot P(s)$$
$$= (2+1) \cdot \frac{4}{9} + (1+1) \cdot \frac{2}{9} + (1+0) \cdot \frac{2}{9} + (0+0) \frac{1}{9}$$
$$= \left(2 \cdot \frac{4}{9} + 1 \cdot \frac{4}{9}\right) + \left(1 \cdot \frac{2}{9} + 1 \cdot \frac{2}{9}\right) + \left(1 \cdot \frac{2}{9} + 0 \cdot \frac{2}{9}\right) + \left(0 \cdot \frac{1}{9} + 0 \cdot \frac{1}{9}\right)$$
$$= 2 \cdot \frac{4}{9} + \cdots + 0 \cdot \frac{1}{9} + 1 \cdot \frac{4}{9} + \cdots + 0 \cdot \frac{1}{9}$$
$$= E[\mathcal{X}] + E[\mathcal{Y}]$$

Theorem:

$$\boxed{E[\mathcal{X} + \mathcal{Y}] = E[\mathcal{X}] + E[\mathcal{Y}]}$$

We have a collection of pairs of shoes, all thrown together.

We randomly form new pairs of left and right shoes.

Let X be the number of correct pairs. What is $E[X]$?

Suppose we have n pairs of shoes.

$$X_i = \begin{cases} 1 & \text{if } i\text{-th pair is correct} \\ 0 & \text{otherwise} \end{cases}$$

$$X_1 + X_2 + \dots + X_n = X$$

$$E[X] = \sum_{i=1}^{n} E[X_i]$$

$$E[X_i] = P[i \text{ is matched correctly}] = \frac{1}{n}$$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Permutation of degree n: bijective / 1-to-1 mappings
$\{1, \dots, n\} \to \{1, \dots, n\}$

Π_n = set of all permutations of degree n

We count the fixpoints!

$\# \Pi_n = n!$

$X(\pi)$ = # fixpoints of π

$$\Pi_1 = \{ \overset{1}{(1)} \} \qquad \frac{1}{1} = 1$$

$$\Pi_2 = \{ \overset{2}{(1,2)}, \overset{0}{(2,1)} \} \qquad \frac{2}{2} = 1$$

$$\Pi_3 = \{ \overset{3}{(1,2,3)}, \overset{1}{(1,3,2)}, \overset{1}{(2,1,3)}, \overset{0}{(2,3,1)}, \overset{0}{(3,1,2)}, \overset{1}{(3,2,1)} \} \qquad \frac{6}{6} = 1$$

New Problem:

$\boxed{\pi \text{ has an } \underline{\text{upward step}} \text{ at pos } i \text{ if } \underline{\pi(i+1) = \pi(i) + 1}}$

$X(\pi)$ = # of upward steps in π

$E[X] = ?$

$$X_i(\pi) = \begin{cases} 1 & \pi \text{ has an upward step at position } i \\ 0 & \text{otherwise} \end{cases},$$

$1 \le i < n$ (X_i doesn't make sense for last pos.)

$E[X_1]$

$=$ P[π has an upward step at pos 1] $= ?$

P[π has an upward step at pos 1]

$$\frac{\#\text{ perm. with upward step at } 1}{\#\text{ all perm. of deg } n} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

P[π has an upward step at pos 1]

$$\Big(\underset{n-1}{\uparrow},\ \underset{1}{\uparrow},\ \underset{(n-2)!}{\text{rest}} \Big)$$

$E[X_1] = \frac{1}{n}$, $E[X_i] = \frac{1}{n}$, $1 \le i < n$

$$E[X] = E\left[\sum_{i=1}^{n-1} X_i\right] = \sum_{i=1}^{n-1} E[X_i] = \sum_{i=1}^{n-1} \frac{1}{n} = \frac{n-1}{n}$$

π has an upward jump at pos i if $\pi(i) < \pi(i+1)$

$X(\pi) :=$ # of upward jumps in π

What is $E[X]$?

Exercise!

A Problem from the Textbook

Hence, no matter how many letters there are, on the average, exactly one of the letters will be in its own envelope. ■

EXAMPLE 4.5h Suppose there are 20 different types of coupons and suppose that each time one obtains a coupon it is equally likely to be any one of the types. Compute the expected number of different types that are contained in a set for 10 coupons.

SOLUTION Let X denote the number of different types in the set of 10 coupons. We compute $E[X]$ by using the representation

$$X = X_1 + \cdots + X_{20}$$

where

$$X_i = \begin{cases} 1 & \text{if at least one type } i \text{ coupon is contained in the set of 10} \\ 0 & \text{otherwise} \end{cases}$$

Now

$$\begin{aligned} E[X_i] &= P\{X_i = 1\} \\ &= P\{\text{at least one type } i \text{ coupon is in the set of 10}\} \\ &= 1 - P\{\text{no type } i \text{ coupons are contained in the set of 10}\} \\ &= 1 - \left(\tfrac{19}{20}\right)^{10} \end{aligned}$$

when the last equality follows since each of the 10 coupons will (independently) not be a type i with probability $\frac{19}{20}$. Hence,

$$E[X] = E[X_1] + \cdots + E[X_{20}] = 20\left[1 - \left(\tfrac{19}{20}\right)^{10}\right] = 8.025 \quad ■$$

An important property of the mean arises when one must predict the value of a random variable. That is, suppose that the value of a random variable X is to be predicted.

Abstract Reformulation

Drawing from urn: n balls $1, \dots, n$ k draws

draw: $d = (b_1, b_2, \dots, b_k)$, $b_j \in \{1, \dots, n\}$

$X(d) = \#$ different balls in d

$$X_i(d) = \begin{cases} 1 & \text{at least one } b_j = i\text{, } b_j \text{ occurs in } d \\ 0 & \text{otherwise} \end{cases}$$

$$X(d) = \sum_{i=1}^{n} X_i(d), \qquad X = \sum_{i=1}^{n} X_i$$

Example: $d = (3, 7, 2, 3, 1, 3, 7)$ $n = 10$, $k = 3$

$$X_1(d) = \begin{cases} 1 & \text{if } 1 \text{ occurs} \\ 0 \end{cases}$$

$$X_2(d) = \begin{cases} 1 & \text{if } 2 \text{ occurs.} \end{cases}$$

$$\begin{aligned} E[X_i] &= P[\text{a sequence } d \text{ contains number } i] \\ &= P[\text{at least one draw with } i] \\ &= 1 - P[i \text{ does not occur}) \\ &= 1 - \left(\frac{n-1}{n}\right)^k \end{aligned}$$

k draws, n balls

sequence w/o i

possibilities

$(\underset{1}{n-1}, n-1, \dots, n-1)$ $\cdot k$

each slot has probability $\frac{n-1}{n}$ to be filled with a number other than i

$$X = \sum_{i=1}^{n} X_i \Rightarrow E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1 - \left(\frac{n-1}{n}\right)^k$$