Office Hour, 23 November 2021

Revision of

- Random Variables
- Discrete and continuous RVs
- Cumulative Distribution Function (cdf)
  Verteilungsfunktion
- Probability Mass Function (pmf)
  Wahrscheinlichkeitsfunktion
- Probability Density Function (pdf)
  Wahrscheinlichkeitsdichte
- Uniform and Exponential Distribution
  Gleichverteilung und Exponentialverteilung

Random Variables

Sample space S, elements are outcomes

On (some) subsets we have a probability measure

Subsets $E \subseteq S$ (i.e., that is measurable) are called events.

P maps events to numbers in $[0,1]$

i.e., we have $0 \leq P(E) \leq 1$.

P satisfies Axioms 1, 2, 3.

Examples:

- Throwing 4 dice, $S = \{1, \dots, 6\}^4$
- Tossing a coin infinitely many times

- each such sequence of tosses is of the form

$H, T, T, H, \dots \dots$

We want to abstract from the complex sample spaces

Only look at functions

$$X : S \to \mathbb{R}$$

like:

$X =$ Points of 2nd die,

Sum of points of first three dice

$X =$ number of tosses until first head.

Look at events formulated in terms of such X

$[3 < X \leq 5]$, $[X$ is even$]$, ...

are events, i.e., subsets of S.

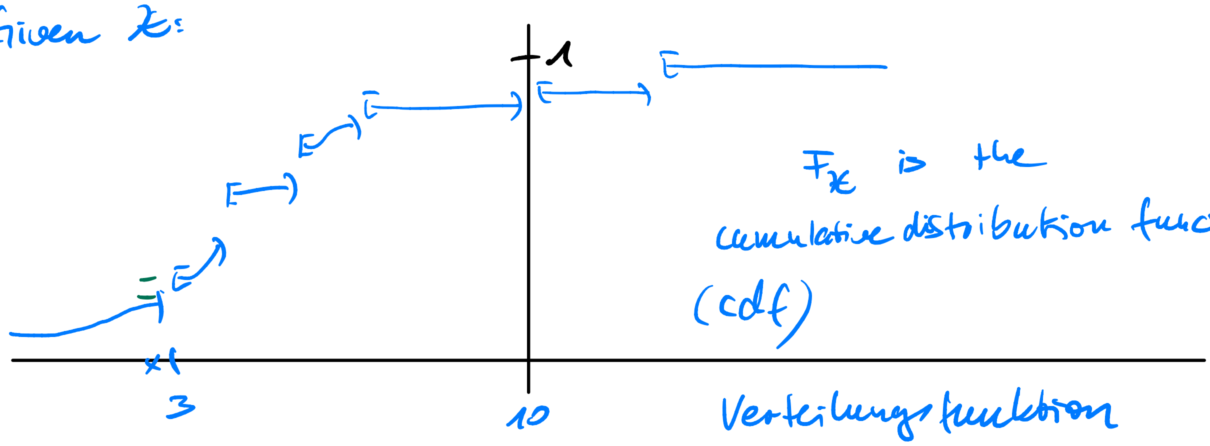
What kind of info about the X can we have?

Since there is prob. P on S, we have

$$P[X \text{ is even}] \quad \text{or} \quad P[3 < X \leq 5]$$

What is the possible structure of the probabilities of events that can be formulated with X?

Given X:



F_X is the cumulative distribution function (cdf)

Verteilungsfunktion

$F_X(x) = P[X \leq x]$

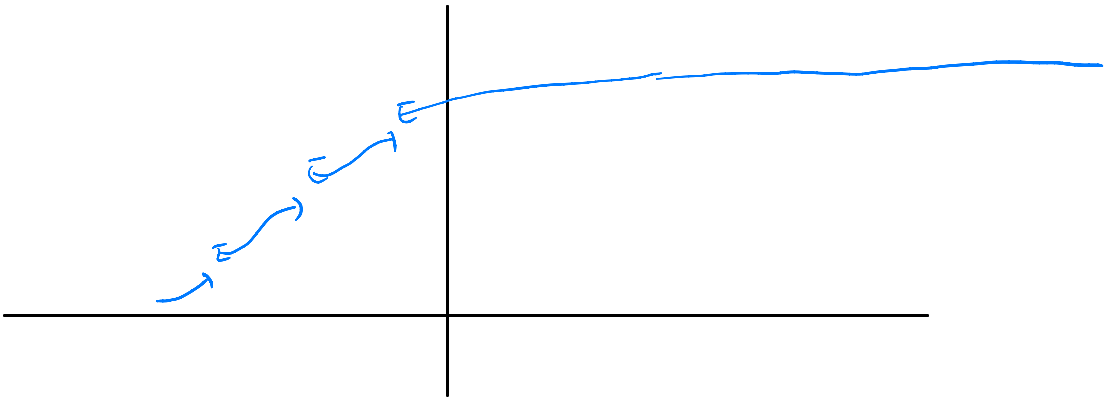
$P[3 < X \leq 4 \text{ or } X > 5]$

$= \underbrace{F_X(4) - F_X(3)} + (1 - F_X(5))$

$P[3 \leq X]$

$= 1 - P[X < 3]$

$= 1 - \lim_{x \to 3} F_X(x)$

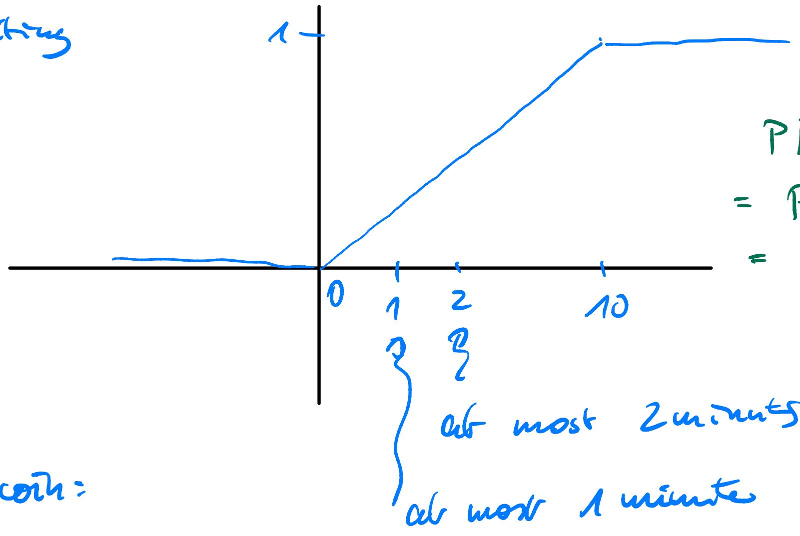


Strange distributions like this practically never occur.

continuous RV

Bus waiting time

1 3 7 3 8 1 4 . . .
usecs



$P[1 \le X \le 2]$

$= P[X \le 2] - P[X < 1]$

$= F_X(2) - \lim_{\substack{x \to 1- \\ x<1}} F_X(x)$

$= F_X(2) - F_X(1)$

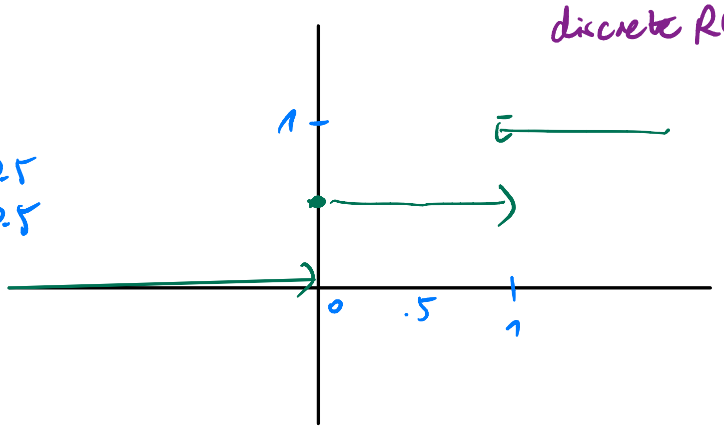
Tossing coin:

$T = 0, \; H = 1$

$P[H] = 0.5$

$X \in \{0, 1\}, \quad P[X=0] = 0.5$
$\quad P[X=1] = 0.5$

discrete RV



$F_X(x) = P[X \le x]$

Given a distribution function, how can we calculate probabilities:

- $P[a < X \le b] = F_X(b) - F_X(a)$

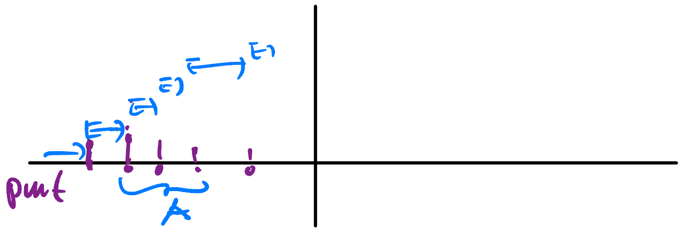
Other possibilities for (i) discrete or (ii) continuous RVs

X = rolling a die (1st die out of 1 million dice)

$P[X \text{ is even}] = P[X=2] + P[X=4] + P[X=6]$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

X is discrete iff X takes on only finitely or countably many values



cdf of a discrete RV

All we need to calculate probabilities about X are the p.s $P[X = x_i]$, x_i possible value of X

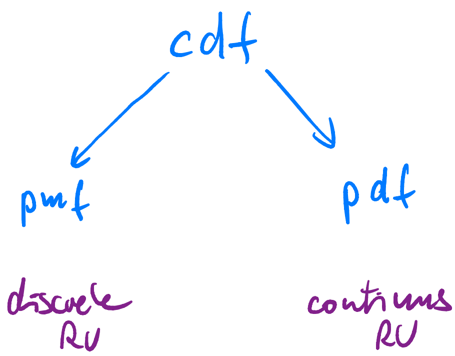
$\Updownarrow$

probability mass function

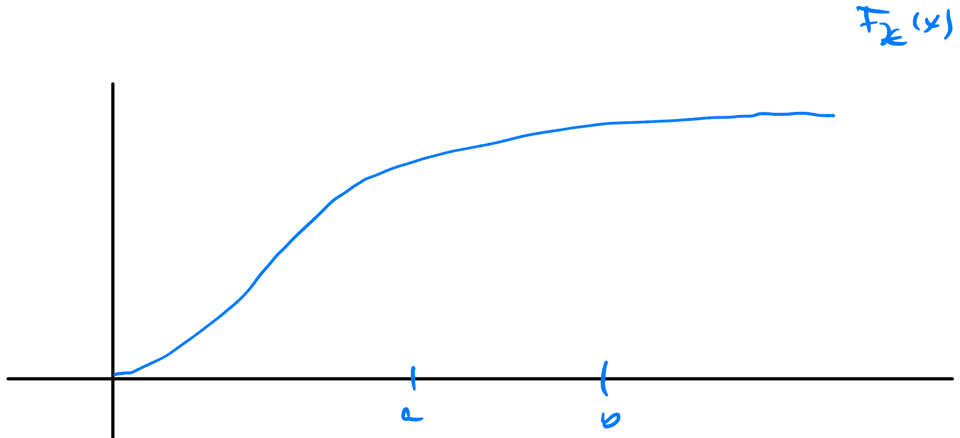
Wahrscheinlichkeitsfunktion

let $x_1, x_2, \ldots$ be the possible values of X

let $A \subseteq \mathbb{R}$. $P[X \in A] = \sum_{x_i \in A} p(x_i)$



probability density function



$$P[X \in [a,b]] = F_X(b) - F_X(a)$$

$$= \int_a^b f(x)\,dx$$

probability density fct of X

let $f_X = F_X'$

$f = F'$ $\quad f$ is the derivative of F

F is the antiderivative of f

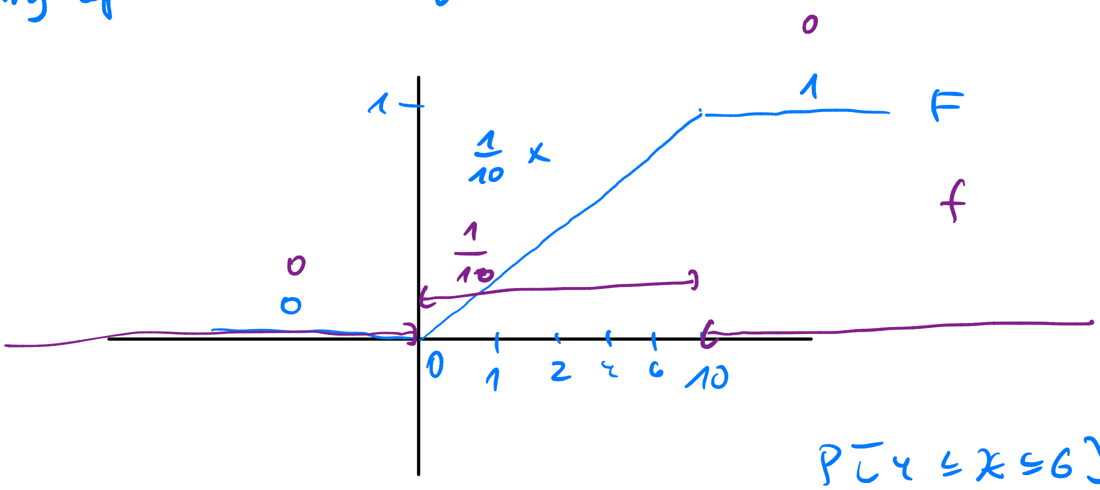
We can also compute probabilities of X using f

$A = [a_1, b_1) \cup (a_2, b_2)$

$\Rightarrow$

$$P[X \in A] = \int_A f(x)\,dx$$

Density of bus waiting time



$$f(x_0) = F'(x_0) = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h}$$

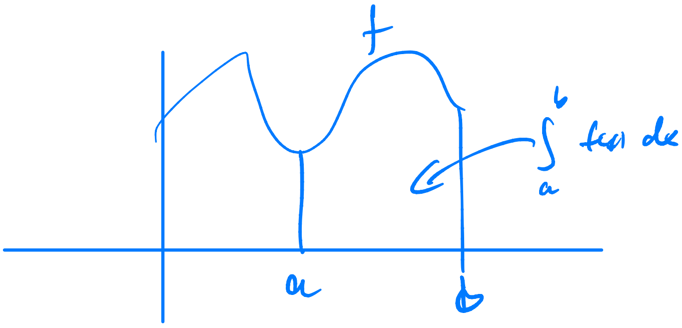
$$= \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

$$P[4 \le X \le 6]$$

$$= \int_4^6 f(x)\,dx$$

$$= 2 \cdot \frac{1}{10}$$

Why integral



Area under the curve

Exponential distribution:

X is exponentially distributed

X is a waiting time for atomic decay

$$P[\underbrace{X > s+t}_{\mathcal{E}} \mid \underbrace{X > s}_{\mathcal{F}}] = P[X > t]$$

Story of the process w/o memory

$s,t \geq 0$

$$\parallel \quad \mathcal{E} \wedge \mathcal{F}$$

$$\frac{P[X > s+t \wedge X > s]}{P[X > s]} \quad \mathcal{F}$$

$$\frac{P[X \overset{\mathcal{E}}{>} s+t]}{P[X > s]} = P[X > t]$$

$$\Leftrightarrow \quad P[X > s+t] = P[X > s]\,P[X > t]$$

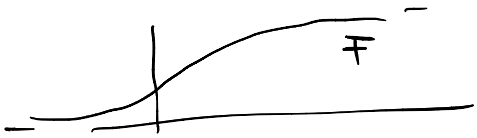
$$G(s+t) = G(s) \cdot G(t)$$

What kind of function satisfies

$$G(s+t) = G(s)\cdot G(t)$$

$$a^{s+t} = a^s \cdot a^t$$

This is the only kind of continuous fct satisfying this equation $G(s) = a^s$



$$G(s) = P[X > s] = 1 - P[X \leq s] = 1 - F(s)$$

$0 < a < 1$

$$a^s = \left(e^{\log a}\right)^s = \left(e^{-\lambda}\right)^s = e^{-\lambda s}$$

$\log a < 0$

$\lambda := -\log a$

$\Rightarrow \lambda > 0$

$$F(s) = 1 - G(s) = 1 - a^s$$

$\Rightarrow 0 < a < 1$

$$F(s) = 1 - G(s) = 1 - a^s \quad \text{for some } a,\ 0<a<1$$

$$= 1 - e^{-\lambda s} \quad \text{—} \quad \lambda,\ \lambda > 0$$

$$f(s) = F'(s) = \lambda e^{-\lambda s}$$

$$\left(e^{-\lambda x}\right)'$$

$$g(h(x))' = g'(h(x))\cdot h'(x) = e^{-\lambda x}\cdot(-\lambda) = -\lambda e^{-\lambda x}$$

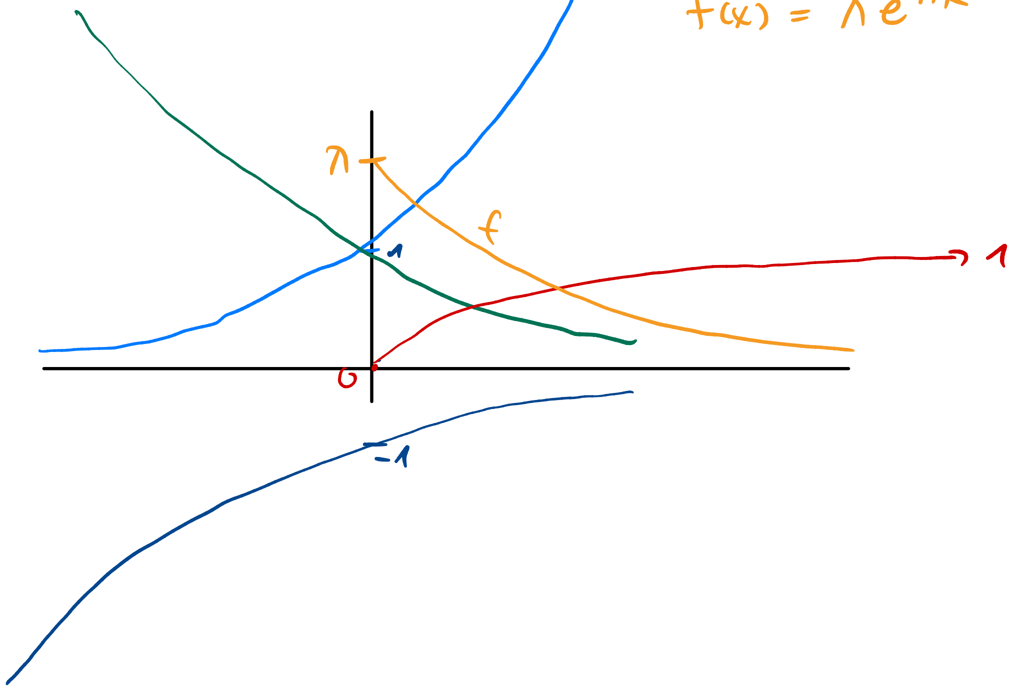
$h(x) = -\lambda x \Rightarrow h'(x) = -\lambda$

$g(y) = e^y \Rightarrow g'(y) = e^y$

Pictures

$F(x) = 1 - e^{-\lambda x}$

$f(x) = \lambda e^{-\lambda x}$



$e^{\lambda x}$

$e^{-\lambda x}$

$-e^{-\lambda x}$

$1 - e^{-\lambda x}$

Functions can be combined, $g \circ f$

$\sin(\sqrt{x})$

$(\sin \circ \sqrt{\cdot})(x)$

$\log = \exp^{-1}$