Rules for Mean/Expected Value

- $E[a] = a$
- $E[aX] = a\,E[X]$
- $E[X+Y] = E[X] + E[Y]$

Rules for Variance

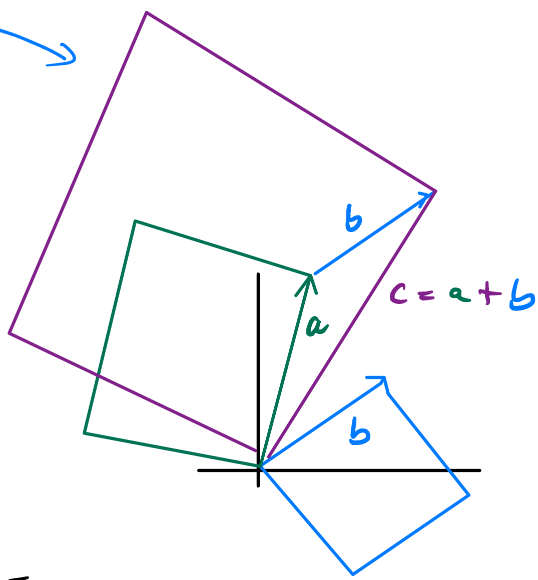
- $Var(a) = 0$
- $Var(aX) = a^2\,Var(X^2)$
- $Var(X+Y) = Var(X) + Var(Y)$

  iff "$X-\mu_X$ and $Y-\mu_Y$ are meeting at a right angle"

Note that variance involves squaring:

$$Var(X) = \sum_i (x_i - \mu_X)^2\, p(x_i)$$

Let's look at squares in the plane (suppose that □, □, and □ in the figure are squares):



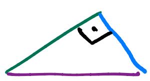
□ is the square with vector a as one side

□ has b as one side

□ has $c = a + b$ as one side.

Then □ + □ = □

iff

the triangle  is rectangular

Back to linear algebra

$\bar{x} = (x_1, \dots, x_n)$

$\bar{y} = (y_1, \dots, y_n)$

$\bar{x}$ and $\bar{y}$ are orthogonal iff $\bar{x} \cdot \bar{y} = 0$

Now we are looking at

$\mathcal{X} - \mu_{\mathcal{X}}$, $\mathcal{Y} - \mu_{\mathcal{Y}}$

It turns out that

$$\mathrm{Var}(\mathcal{X} + \mathcal{Y}) = \mathrm{Var}(\mathcal{X}) + \mathrm{Var}(\mathcal{Y})$$

iff $\int (x - \mu_{\mathcal{X}})(y - \mu_{\mathcal{Y}})\, d(x,y) = 0$

iff $\mathrm{Cov}(\mathcal{X}, \mathcal{Y}) = 0$

Cov for discrete RVs: $\sum_{i,j} (x_i - \mu_{\mathcal{X}})(y_j - \mu_{\mathcal{Y}})\, p(x_i, y_j) = 0$

We have:

$$\mathcal{X}, \mathcal{Y} \text{ independent} \Rightarrow \mathrm{Cov}(\mathcal{X}, \mathcal{Y}) = 0$$

Distributive Law

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

$$(a_1 + a_2 + a_3) \cdot (b_1 \cdot b_2)$$

$$= a_1 b_1 + a_1 \cdot b_2 + \dots \quad a_3 b_3$$

inherited by covariance $\mathrm{Cov}(X, Y)$

How did we get X_0?

$$X_0 = \frac{1}{\sigma_X} \cdot X \quad !$$

Idea: Normalize the Variance to 1 by division by the standard deviation

where

$$\sigma_X = \sqrt{\sum_i (x_i - \mu_X)^2 \, p(x_i)}$$

$$\sigma_X = \sqrt{\int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) \, dx}$$

is the standard deviation of X

Variance of Averages

RVs $X_1, \dots, X_n$ independent, identical distribution

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$\overline{X}_n$ is the average of the X_i

$$\mathrm{Var}(\overline{X}_n) = \mathrm{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var}(X_1)$$

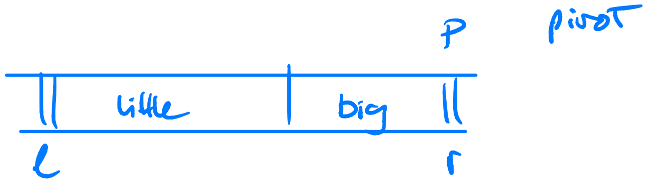
$$= \frac{1}{n^2} n \cdot \mathrm{Var}(X_1) = \frac{1}{n} \mathrm{Var}(X_1)$$

The variance of the n-fold average is $\frac{1}{n}$ · times the variance of the original distribution

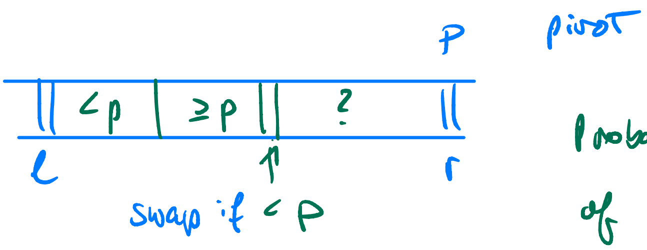
$$\sigma_{\overline{X}_n} = \frac{1}{\sqrt{n}} \sigma_{X_1}$$

Partitioning in Quicksort

Goal

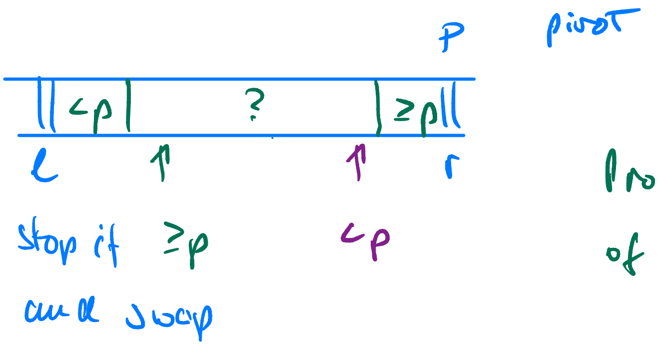


Lomuto-Algorithm



Probability .5 of a swap

Hoare Algorithm



Probability .25 of a swap

Counting the Average Number of Swaps

In our little implementation, we run multiple executions of

- Lomuto Partitioning
- Naive Outside-in Partitioning
- Hoare Partitioning

and count the number of swaps.

We see: The more executions are combined for an average, the less variation is there among the results.