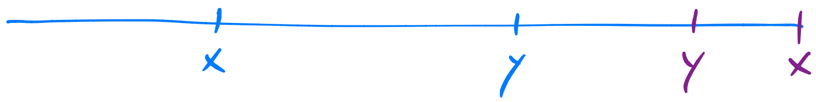
PTS Lecture Notes

Thu, 11 Nov 2021

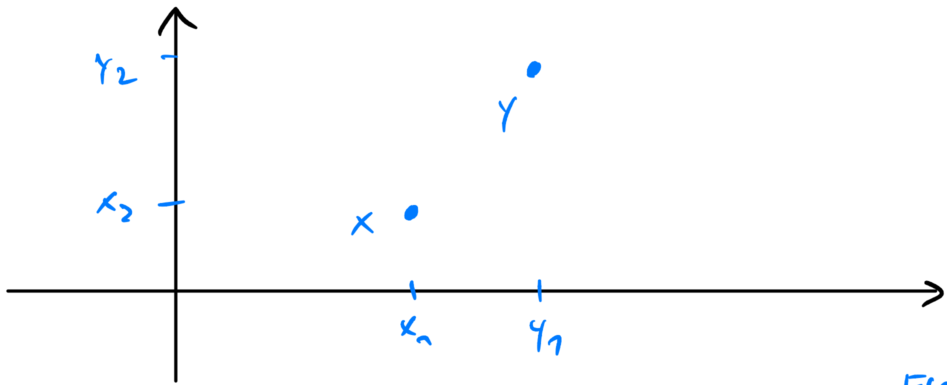
How measure distances?

1.) Points on a line



$$\mathrm{dist}(x,y) = |x-y|$$

2.) Points in the plane



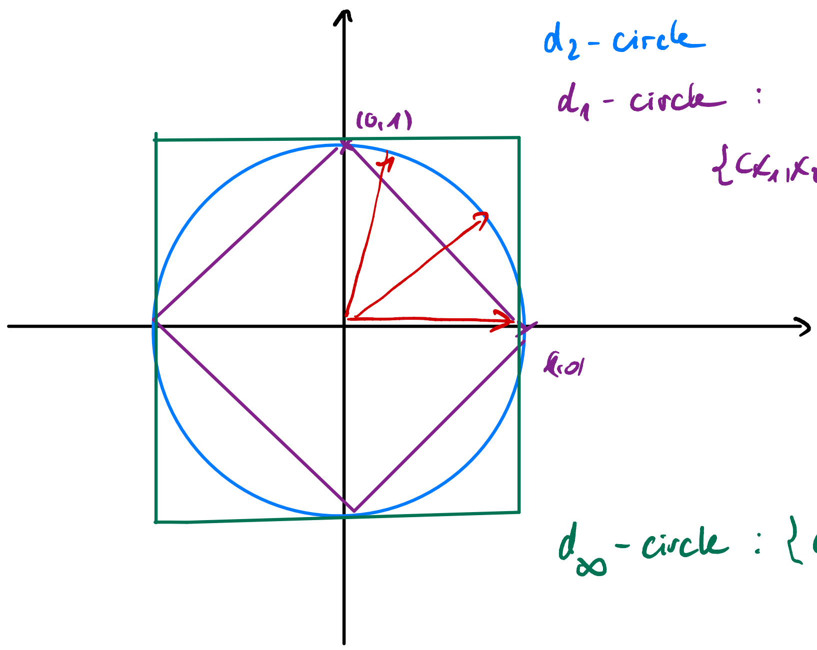
$$\mathrm{dist}_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$ Euclidean distance

Other ways:

$$\mathrm{dist}_1(x,y) = |x_1-y_1| + |x_2-y_2|$$

$$\mathrm{dist}_\infty(x,y) = \sup\{|x_1-y_1|, |x_2-y_2|\}$$

Differences between distances: circles of radius 1



d_2-circle

d_1-circle: $\{(x_1,x_2) \mid |x_1|+|x_2| = 1\}$

d_∞-circle: $\{(x_1,x_2) \mid \max\{|x_1|,|x_2|\} = 1\}$

What happens if we have a vector $\rightarrow$ and rotate it?

Note: d_2 is the only distance that keeps lengths/distances invariant under rotation.

let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$\mu = \frac{1}{n}(x_1 + \dots + x_n)$ (like mean in a uniform distributions)

$m = (\mu, \dots, \mu) \in \mathbb{R}^n$

$$\mathrm{dist}(x, m) = \sqrt{(x_1 - \mu)^2 + \dots + (x_n - \mu)^2}$$

is the distance between x and m

The square of the distance is

$$(\mathrm{dist}(x, m))^2 = (x_1 - \mu)^2 + \dots + (x_n - \mu)^2$$

X RV: Variance of X is

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

Standard deviation of X (Standard abweichung)

$$\sigma = \sigma_X = \sqrt{\mathrm{Var}(X)} \implies \mathrm{Var}(X) = \sigma^2$$

What is that?

Assume: X has values $x_1, \dots, x_n$, with $p(x_i) = p_i$

Mean μ $\left(= \sum_i x_i \cdot p(x_i)\right)$

$$\mathrm{Var}(X) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

Similar to d_2-distance, but balls are ellipsoids

$$\sigma_X = \sqrt{\sum_i (x_i - \mu)^2 \cdot p(x_i)}$$

Weighted Euclidean distance of the possible values from μ

Varianz

X diskret

$$Var(X) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

$X \sim f$

$$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\sigma_X = \sqrt{\sum_i (x_i - \mu)^2 \cdot p(x_i)}$$

$$\sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx}$$

Wie kann man die Varianz einfach berechnen?

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + E[\mu^2] \\ &= E[X^2] - 2\mu \cdot \mu + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

Beispiel: Würfelzahl W

Wir wissen $E[W] = \frac{7}{2}$ $\quad E[W^2] = \frac{91}{6}$

$$Var(W) = E[W^2] - E[W]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{2 \cdot 91 - 49 \cdot 3}{12} = \frac{182 - 147}{12} = \frac{35}{12}$$

Stetige Verteilung: Exponentielle V. mit Dichte $\lambda \cdot e^{-\lambda x}$, $x \geq 0$

$X \sim \lambda \cdot e^{-\lambda x}$,

$E[X] = \frac{1}{\lambda}$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_0^{\infty} \underbrace{x^2}_{u} \cdot \underbrace{\lambda e^{-\lambda x}}_{v'} dx$$

$$= \Big[\underbrace{x^2}_{u} \cdot \underbrace{(- e^{-\lambda x})}_{v} \Big]_0^{\infty} - \int_0^{\infty} \underbrace{2x}_{u'} \cdot \underbrace{(- e^{-\lambda x})}_{v} dx$$

$$= 0 - 0 + 2\int_0^{\infty} x \cdot e^{-\lambda x} dx = \frac{2}{\lambda} \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

$$\frac{2}{\lambda} E[X] = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\sigma_X = \frac{1}{\lambda}$$

Eigenschaften der Varianz

- $Var(X+b) = Var(X)$
- $Var(aX) = a^2 \, Var(X)$

$Var(X+Y) = Var(X) + Var(Y)$ falls X, Y unabh.

Wir rechnen: $Y = aX + b$

$$Var(aX+b) = E[Y^2] - E[Y]^2$$

$$= E[(aX+b)^2] - (E[aX+b])^2$$

$$= E[a^2X^2 + 2aXb + b^2] - (a\,E[X] + b)^2$$

$$= a^2 E[X^2] + 2ab\,E[X] + b^2 - (a^2 E[X]^2 + 2ab\,E[X] + b^2)$$

$$= a^2 E[X^2] + \cancel{2ab\,E[X]} \cancel{+b^2} - a^2 E[X]^2 - \cancel{2ab\,E[X]} \cancel{-b^2}$$

$$= a^2 (E[X^2] - E[X]^2) =$$

$$= a^2 \, Var(X)$$

Folgerung

$$\sigma_{aX} = \sqrt{Var(aX)} = \sqrt{a^2 Var(X)}$$

$$= a\sqrt{Var(X)} = a \cdot \sigma_X$$

2.7 Kovarianz

$$Var(X + X) = Var(2X)$$

$$= 4\,Var(X) \neq Var(X) + Var(X)$$

Was macht den Unterschied aus?

$$\mathrm{Var}(X) = E[(X-\mu_X)\cdot(X-\mu_X)]$$

Innere Produkte:

$x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n)$

$\Rightarrow \quad x \bullet y = x_1 y_1 + \dots + x_n y_n$

$\mathrm{dist}(x,y) = \sqrt{(x-y)\cdot(x-y)}$

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)\cdot(Y-\mu_Y)]$$

Speziell:

$\mathrm{Var}(X) = \mathrm{Cov}(X,X)$

$(X,Y) \sim f(x,y)$

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)\cdot(Y-\mu_Y)]$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \underbrace{(x-\mu_X)(y-\mu_Y)}_{\text{Integrand}} \cdot f(x,y) \cdot d(x,y)$$

Integrand

groß: X und Y weichen ab vom Mittelwert

klein: eins von X, Y ist klein

positiv: beide Werte, X und Y haben gleiches Vorzeichen

negativ: beide Werte unterschiedl. Vorzeichen

Eigenschaften der Kovarianz

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y]$$
$$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y$$
$$= E[XY] - \mu_X \mu_Y - \cancel{\mu_Y \mu_X} + \cancel{\mu_X \mu_Y}$$
$$= E[XY] - E[X] \cdot E[Y]$$

Cov. misst den Grad der Abh-

Beobachtung: X, Y unabh.

$$E[XY] = E[X] \cdot E[Y]$$

$$\Rightarrow \mathrm{Cov}(X,Y) = 0$$