# PTS Lecture Notes

Tue, 2 Nov 2021

## Random Variable (RV)

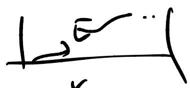
$X : \mathcal{S} \longrightarrow \mathbb{R}$

Distribution fct. (cumulative distribution fct, cdf)

$F_X(x) = F(x) = P[X \le x]$

$F: \mathbb{R} \to [0,1]$, $\lim\limits_{x \to -\infty} F(x) = 0$, $\lim\limits_{x \to \infty} F(x) = 1$,

F is non-decreasing,



$x_0$

$\lim\limits_{x \to x_0+} F(x) = F(x_0)$, $\lim\limits_{x \to x_0-} F(x)$ exists

$p(x) = P[X = x]$ prob. mass fct.
Wkeitsfkt.

Two cases:

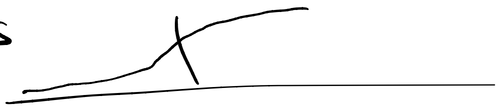
1) $F$ is step fct

$X$ has finitely many (or countably infinitely) values $x_1, \dots, x_n, \dots$ s.th. $p_i = P[X = x_i] > 0$.

and $\sum_{i=0}^{\infty} p_i = 1$ (or $\sum_{i=1}^{n} p_i = 1$)

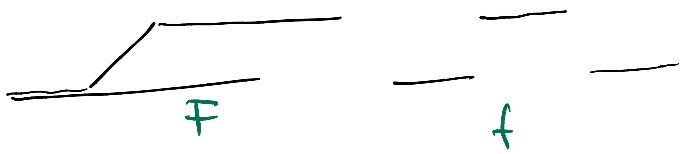
$X$ is discrete (Ex: Sum of points of dice)

2) $F$ is continuous



also: $X$ continuous

No single value $x_0$



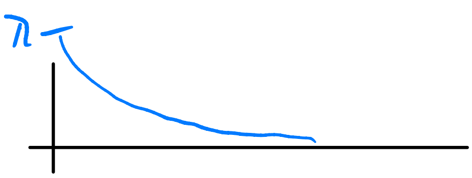

has $P[X = x_0] > 0$.

Analogue to pmf is $f = F'$, called density of $X$

Exp. distr. is continuous

$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}, \quad \lambda > 0$$

Density: $f(x) = F'(x) = \begin{cases} 0 & x \le 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$



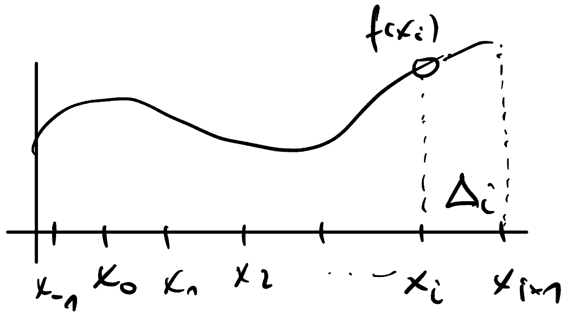
# Expected Value

Intuitively: The long-term average

Ex: 1 dice: , $X$ Points

$$E[X] = 3.5$$

$$\sum_x x \cdot P[X=x] = \sum_i x_i \cdot p_i \leftarrow \text{pmf}$$




$$E[X] \approx \sum_{i=-\infty}^{\infty} x_i \cdot P[x_i \leq X \leq x_{i+1}]$$

$$\approx \sum_{i=-\infty}^{\infty} x_i \cdot f(x_i) \cdot \Delta_i$$

Idea: Weighted avg of possible values, with prob of values as weights $\longrightarrow \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

# Erwartungswert der Exponentialverteilung

Sei $X$ verteilt mit Dichte

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \lambda e^{-\lambda x} & x > 0 \end{cases}$$

$E[X] = ?$

$$= \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{0} x \cdot 0\, dx + \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} x \cdot e^{-\lambda x}\, dx$$

$f(g(x))' = f'(g(x)) \cdot g'(x)$

$f(y) = e^y \Rightarrow f'(y) = e^y$

$g(x) = -\lambda x \Rightarrow g'(x) = -\lambda$

$\int x\, dx = \frac{x^2}{2}$ $\qquad \frac{d}{dx} x^n = n\, x^{n-1}$

$(e^{-\lambda x})' = -\lambda e^{-\lambda x} \Longrightarrow \int e^{-\lambda x}\, dx = -\frac{1}{\lambda} e^{-\lambda x}$

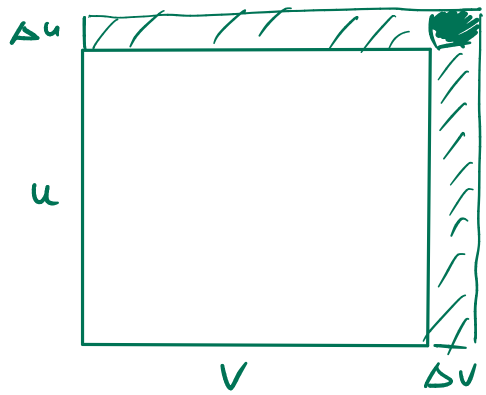
## Partielle Integration : Umkehrung der Produktregel

$$(uv)' = uv' + u'v$$

$$\int (uv)' = \int uv' + \int u'v$$

$$uv = \ldots$$

$$\int uv' = uv - \int u'v$$




$$(u+\Delta u)\cdot(v+\Delta v) - uv$$
$$= uv + \Delta u\cdot v + u\cdot\Delta v + \Delta u\cdot\Delta v - uv$$

$$\lambda \int_0^\infty \underbrace{x}_{u}\cdot \underbrace{e^{-\lambda x}}_{v'}\, dx = \lambda \Big[\underbrace{x}_{u}\cdot\underbrace{\left(-\tfrac{1}{\lambda} e^{-\lambda x}\right)}_{v}\Big]_0^\infty - \lambda\int_0^\infty \underbrace{1}_{u'} \underbrace{\left(-\tfrac{1}{\lambda} e^{-\lambda x}\right)}_{v} dx$$

$$= \lambda[0-0] + \lambda\int_0^\infty \frac{1}{\cancel{\lambda}}\, e^{-\lambda x}\, dx$$

$$= \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty = 0 - \left(-\frac{1}{\lambda}\cdot 1\right) = \frac{1}{\lambda}$$

Physik! Dimensionen wie $\frac{\text{Weg}}{\text{Zeit}}$, $\frac{1}{\text{Zeit}}$ (Rate, Frequenz)

$[x] = \text{Zeit}$, $[\lambda x] =$ ohne Dimension

$[\lambda] = \frac{1}{\text{Zeit}} \Rightarrow \left[\frac{1}{\lambda}\right] = \text{Zeit}$

$x$ war Wartezeit $\Rightarrow \frac{1}{\lambda}$ = ⌀liche Wartezeit

Etwa: $\lambda = 10$, d.h. 10 Zerfälle je Stunde

$\Rightarrow$ Erwartungswert der Wartezeit $= \frac{1}{10}$ Stunde

## 2.2. Gemeinsame Verteilungen

Wir betrachten zwei ZV zusammen

- Größe, Gewicht bei Personen
- Niederschlag (precipitation) und Temperatur
- Messfehler in x- und y-Richtung

betrachten W.keiten der Art

$$P[X = x, Y = y]$$

$$P[a < X \le b, c < Y \le d]$$

Beispiel 29: 9 Batterien: 2 neu, 3 teilweise geladen, 4 leer

Wähle zufällig 3 der 9 Batterien

$X$ = # neue Batterien $\qquad X = \{0, 1, 2\}$

$Y$ = # teilw. gel. Batterien $\qquad Y = \{0, 1, 2, 3\}$

$X, Y$ diskret! $\quad P[X = x, Y = y] =: p(x,y)$ gemeinsame W.keitsfkt. von $X$ und $Y$

$$p(0,0) = \frac{\binom{2}{0}\binom{3}{0}\binom{4}{3}}{\binom{9}{3}} = \frac{\binom{4}{1}}{\frac{\overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot 7}{\cancel{3} \cdot \cancel{2} \cdot 1}} = \frac{4}{84}$$

$$p(0,1) = \frac{\binom{3}{1}\binom{4}{2}}{84} = \frac{3 \cdot 6}{84} = \frac{18}{84}$$

$$p(0,2) = \frac{\binom{3}{2}\binom{4}{1}}{84} = \frac{3 \cdot 4}{84} = \frac{12}{84}$$

$$p(0,3) = \frac{\binom{3}{3}}{84} = \frac{1}{84}$$

$$p(1,0) = \frac{\binom{2}{1}\binom{3}{0}\binom{4}{2}}{84} = \frac{2 \cdot 1 \cdot 6}{84} = \frac{12}{84}$$

$$p(1,1) = \frac{\binom{2}{1}\binom{3}{1}\binom{4}{1}}{84} = \frac{2 \cdot 3 \cdot 4}{84} = \frac{24}{84}$$

$$p(1,2) = \frac{\binom{2}{1}\binom{3}{2}\binom{4}{0}}{84} = \frac{2 \cdot 3 \cdot 1}{84} = \frac{6}{84}$$

$$p(2,0) = \frac{\binom{2}{2}\binom{3}{0}\binom{4}{1}}{84} = \frac{1 \cdot 1 \cdot 4}{84} = \frac{4}{84}$$

$$p(2,1) = \frac{\binom{2}{2}\binom{3}{1}\binom{4}{0}}{84} = \frac{1 \cdot 3 \cdot 1}{84} = \frac{3}{84}$$

Wir haben die gem. W.keitsfunktion von $X$ und $Y$

Zusammenfassung

| $X \backslash Y$ | 0 | 1 | 2 | 3 | Summe: $P[X=x]$ |
|---|---|---|---|---|---|
| 0 | $4/84$ | $18/84$ | $12/84$ | $1/84$ | $35/84$ |
| 1 | $12/84$ | $24/84$ | $6/84$ | 0 | $42/84$ |
| 2 | $4/84$ | $3/84$ | 0 | 0 | $7/84$ |
| Summe $P[Y=y]$ | $20/84$ | $45/48$ | $18/84$ | $1/84$ | |

gem. W.keitsfkt. von $X$ und $Y$

Randwahrscheinlichkeiten

marginal probabilities

marginale W.keitsfunktionen


| $X \backslash Y$ | 0 | 1 | 2 | 3 | |
|---|---|---|---|---|---|
| 0 | $4/84$ | $18/84$ | $12/84$ | $1/84$ | $35/84$ |
| 1 | $12/84$ | $24/84$ | $6/84$ | 0 | $42/84$ |
| 2 | $4/84$ | $3/84$ | 0 | 0 | $7/84$ |
| | $20/84$ | $45/48$ | $18/84$ | $1/84$ | |


Wahrscheinlichkeitsfkt.

gem. Verteilung

$P[X \leq x, Y \leq x]$

| $X \backslash Y$ | $<0$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $<0$ | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | $4/84$ | $22/84$ | $34/84$ | $35/84$ |
| 1 | 0 | $16/84$ | $58/84$ | $76/84$ | $77/84$ |
| 2 | 0 | $20/84$ | $65/84$ | $83/84$ | $84/84$ |

## Marginale W.keitsfkt.

$$p(x,y) = P[\mathcal{X}=x, \mathcal{Y}=y]$$

$$p_{\mathcal{X}} = P[\mathcal{X}=x] = \sum_{j=1}^{n} P[\mathcal{X}=x, \mathcal{Y}=y_j]$$

$$= \sum_{j=1}^{n} p(x, x_j)$$

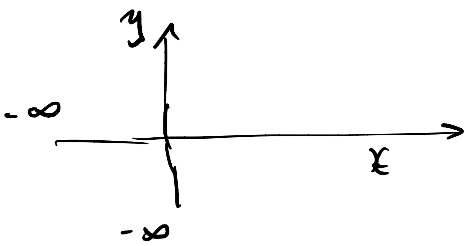
$$p_{\mathcal{Y}} = P[\mathcal{Y}=y] = \sum_{i=1}^{m} p(x_i, y)$$

Seien $\mathcal{X}, \mathcal{Y}$ diskret mit gem. W.keitsfkt.

$$p(x,y) = P[\mathcal{X}=x, \mathcal{Y}=y]$$

Die gemeinsame Verteilung von $\mathcal{X}$ und $\mathcal{Y}$ ist

$$F: \mathbb{R} \times \mathbb{R} \to [0,1], \quad F(x,y) = P[\mathcal{X} \le x, \mathcal{Y} \le y]$$




Die marginale Verteilung von $\mathcal{X}$ ist

$$F_{\mathcal{X}}(x) = P[\mathcal{X} \le x] = P[\mathcal{X} \le x, \mathcal{Y} < \infty]$$

$$= \lim_{y \to \infty} F(x,y)$$

# Gemeinsame Dichte stetiger ZVen

$X, Y$ stetig

Dichte $\quad f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad f \geq 0$

Für „vernünftige" $C \subseteq \mathbb{R} \times \mathbb{R}$ gilt

$$P[(X, Y) \in C] = \iint_{(x,y) \in C} f(x,y) \, d(x,y)$$

Volumen unter der Fläche von $f$ über $C$