Lemma 1: Let $f_1(n) = \Theta(f_2(n))$ and $g_1(n) = \Theta(g_2(n))$.

Then $f_1(n) = o(g_1(n)) \iff f_2(n) = o(g_2(n))$

Proof: "$\Longrightarrow$"

(i) $f_1 = \Theta(f_2) \Rightarrow f_1 = \Omega(f_2) \Rightarrow f_2 = O(f_1)$

(ii) $g_1 = \Theta(g_2) \Rightarrow g_1 = O(g_2)$

(iii) $f_1 = o(g_1)$

Hence, (i) $\wedge$ (iii) $\wedge$ (ii) $\Rightarrow$ $f_2 = o(g_2)$

[Notice that we can conclude:

- $f = O(g),\ g = o(h) \Rightarrow f = o(h)$
- $f = o(g) \iff g = \omega(f)$
- $f = \Theta(g) \iff g = \Theta(f)$.

Notice also that

- $o(g), \Omega(g)$ should be considered as classes of functions and that $f = o(g), \Omega(g)$ better be read as $f \in o(g), \Omega(g)$
- $\Theta(g) = O(g) \cap \Omega(g)$]

"$\Leftarrow$" Analogously, we see

(i) $f_1 = \Theta(f_2) \Rightarrow f_1 = O(f_2)$

(ii) $f_2 = O(g_2)$

(iii) $g_1 = \Theta(g_2) \Rightarrow g_1 = \Omega(g_2) \Rightarrow g_2 = O(g_1)$

Hence, (i) $\wedge$ (ii) $\wedge$ (iii) $\Rightarrow f_1 = O(g_1)$

Lemma 1A: Let $f(x), g(x) > 0$. Then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c > 0 \implies f(n) = \Theta(g(n))$$

Proof: Consider some $\varepsilon > 0$ such that $c > \varepsilon$ (and hence $c - \varepsilon > 0$). From the definition of the limit we conclude that there is some $x_0 > 0$ such that for all $x \geq x_0$ we have

$$(*) \qquad \left| \frac{f(x)}{g(x)} - c \right| < \varepsilon$$

From $(*)$ we conclude

$$| f(x) - c \cdot g(x) | < \varepsilon \cdot g(x)$$

$$\implies f(x) - c \cdot g(x) < \varepsilon \cdot g(x)$$

$$\implies f(x) < (c + \varepsilon) \cdot g(x).$$

Similarly, we conclude

$$c - \frac{f(x)}{g(x)} < \varepsilon$$

$$\implies c - \varepsilon < \frac{f(x)}{g(x)}$$

$$\implies (c - \varepsilon)\, g(x) < f(x)$$

$$\implies g(x) < \frac{1}{c - \varepsilon} \cdot f(x)$$

$\square$

Lemma 2: Let $f(x), g(x) > 0$. If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$,

then

- $f(n) = O(g(n))$
- $g(n) \neq O(f(n))$

Proof: The first part is straightforward. To see the second part, let us write up formally what it means:

$$(*) \quad \forall c > 0, \ \forall n_0 \in \mathbb{N}, \exists n_1 \geq n_0 : g(n_1) > c \cdot f(n_1)$$

Because of the limit assumption, we know that for every $c > 0$ there is a number $n_c \in \mathbb{N}$ such that for all $n \geq n_c$ we have

$$\frac{f(n)}{g(n)} < \frac{1}{c}, \quad \text{that is,} \quad g(n) > c \cdot f(n).$$

To satisfy $(*)$, we have to find a suitable n_1, given some $c > 0$ and $n_0 \in \mathbb{N}$. We find this by letting $n_1 := \max\{n_0, n_c\}$.

$\square$

<u>Lemma 3:</u> Let $f(n), g(n), h(n) > 0$ such that

- $\lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \lim_{n \to \infty} h(n) = \infty$

- $f(n) \geq g(n)\, h(n)$.

Then $f(n) \neq O(g(n))$

<u>Proof:</u> Exercise

Reminder

L'Hôpital's Rule (special case)

If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$ and

$\lim_{x \to \infty} \dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$