Data Structures and Algorithms

Chapter 2

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Acknowledgments

- The course follows the book “Introduction to Algorithms‘”, by **Cormen, Leiserson, Rivest and Stein**, MIT Press [CLRST]. Many examples displayed in these slides are taken from their book.
- These slides are based on those developed by Michael Böhlen for this course.

(See http://www.inf.unibz.it/dis/teaching/DSA/)

- The slides also include a number of additions made by Roberto Sebastiani and Kurt Ranalter when they taught later editions of this course

(See http://disi.unitn.it/~rseba/DIDATTICA/dsa2011_BZ//)

DSA, Chapter 2: Overview

- Complexity of algorithms
- Asymptotic analysis
- Correctness of algorithms
- Special case analysis

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Analysis of Algorithms

- Efficiency:
  - Running time
  - Space used
- Efficiency is defined as a function of the input size:
  - Number of data elements (numbers, points)
  - The number of bits of an input number

The RAM Model

We study complexity on a simplified machine model, the RAM (= Random Access Machine):

- accessing and manipulating data takes a (small) constant amount of time

Among the instructions (each taking constant time), we usually choose one type of instruction as a characteristic operation that is counted:

- arithmetic (add, subtract, multiply, etc.)
- data movement (assign)
- control flow (branch, subroutine call, return)
- comparison

Data types: integers, characters, and floats

Analysis of Insertion Sort

Running time as a function of the input size (exact analysis)

```
                                    cost  times
for j := 2 to n do                  c1    n
  key := A[j]                       c2    n-1
  // Insert A[j] into A[1..j-1]
  i := j-1                          c3    n-1
  while i>0 and A[i]>key do         c4    Σ_{j=2}^{n} t_j
    A[i+1] := A[i]                  c5    Σ_{j=2}^{n} (t_j - 1)
    i--                             c6    Σ_{j=2}^{n} (t_j - 1)
  A[i+1]:= key                      c7    n-1
```

t_j is the number of times the while loop is executed, i.e.,

$(t_j - 1)$ is number of elements in the initial segment greater than `A[j]`

Analysis of Insertion Sort/2

- The running time of an algorithm for a given input is the sum of the running times of each statement.
- A statement
  - with cost *c*
  - that is executed *n* times

  contributes *c*n* to the running time.
- The total running time *T(n)* of insertion sort is

$$T(n) = c1*n + c2*(n-1) + c3*(n-1) + c4 * \sum_{j=2}^{n} t_j$$

$$+ c5 \sum_{j=2}^{n} (t_j - 1) + c6 \sum_{j=2}^{n} (t_j - 1) + c7*(n - 1)$$

Analysis of Insertion Sort/3

- The running time is not necessarily equal for every input of size *n*
- The performance depends on the details of the input (not only length *n*)
- This is modeled by t_j
- In the case of Insertion Sort, the time t_j depends on the original sorting of the input array

Performance Analysis

- Often it is sufficient to count the number of iterations of the core (innermost) part
  - no distinction between comparisons, assignments, etc (that means, roughly the same cost for all of them)
  - gives precise enough results
- In some cases the cost of selected operations dominates all other costs.
  - disk I/O versus RAM operations
  - database systems

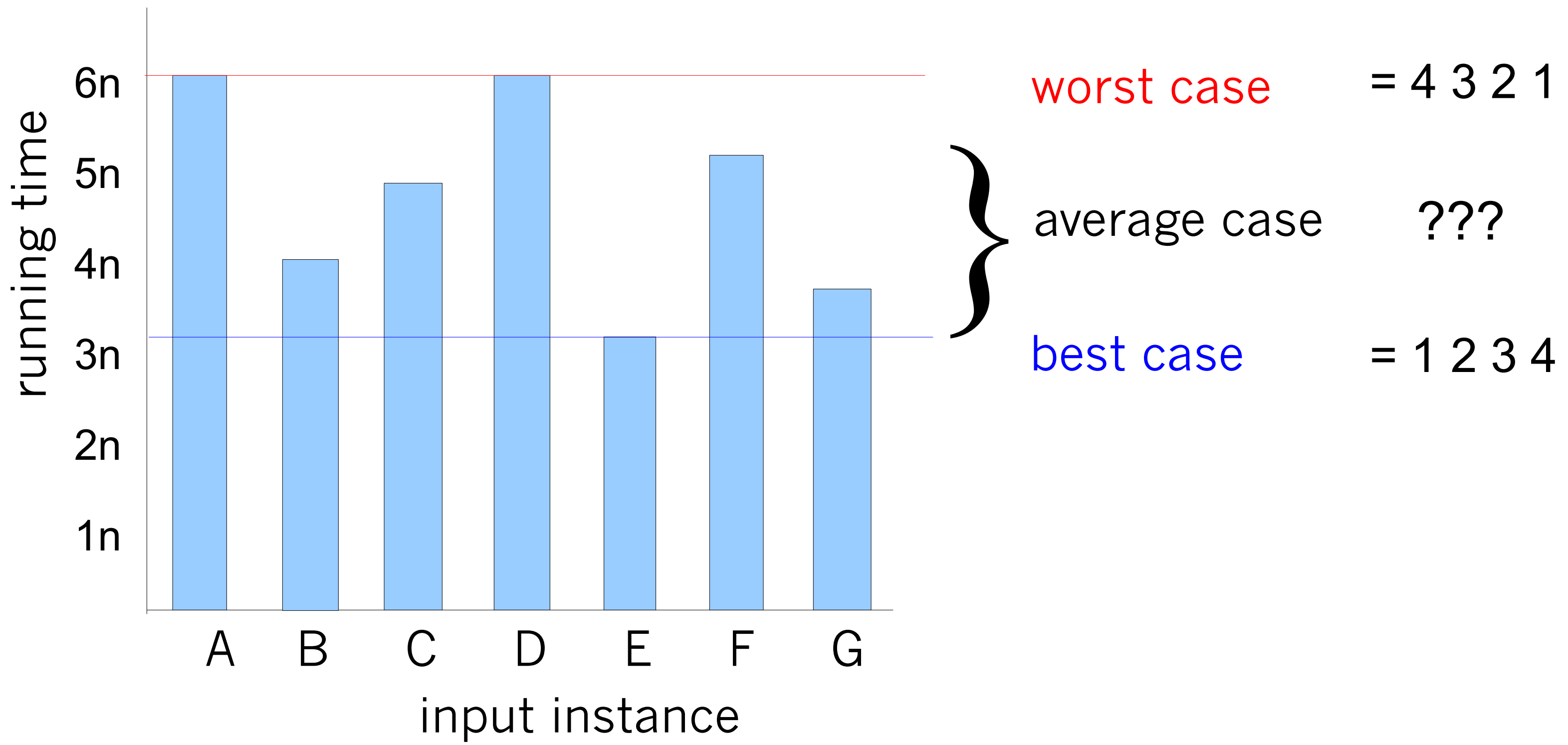
Worst/Average/Best Case

- Analyzing Insertion Sort's
  - Worst case: elements sorted in inverse order, $t_j=j$, total running time is *quadratic* (time = an^2+bn+c)
  - Average case (= average of all inputs of size n): $t_j=j/2$, total running time is *quadratic* (time = an^2+bn+c)
  - Best case: elements already sorted, $t_j=1$, innermost loop is never executed, total running time is *linear* (time = $an+b$)
- How can we define these concepts formally?

  … and how much sense does "best case" make?

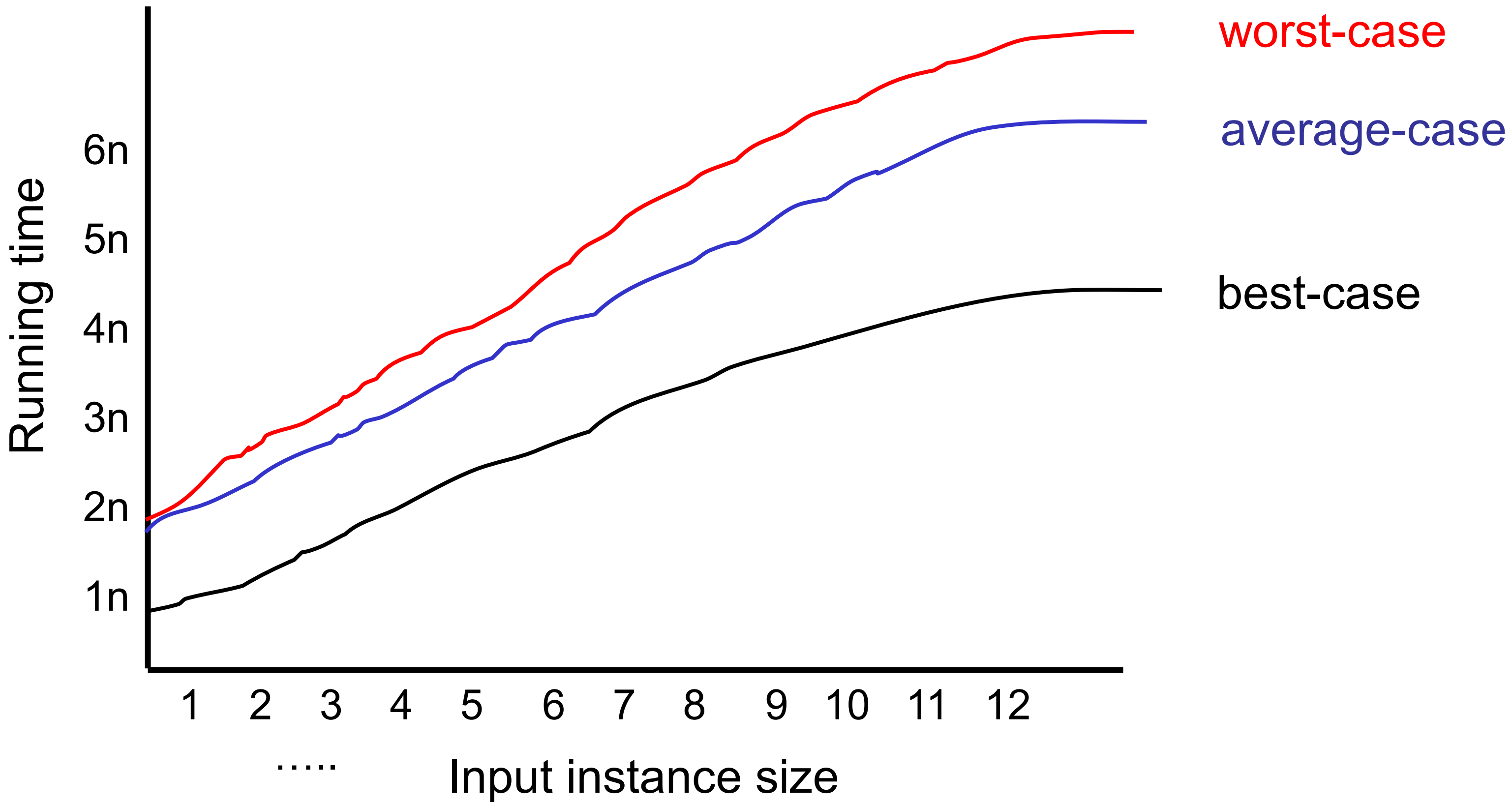
Worst/Average/Best Case/2

For a specific size of input size *n*, investigate running times for different input instances:



Worst/Average/Best Case/3

For inputs of all sizes:



Best/Worst/Average Case/4

Worst case is most often used:

- It is an upper-bound
- In certain application domains (e.g., air traffic control, surgery) knowing the worst-case time complexity is of crucial importance
- For some algorithms, worst case occurs fairly often
- The average case is often as bad as the worst case

The average case depends on assumptions

- What are the possible input cases?
- What is the probability of each input?

Analysis of Linear Search

```
INPUT: A[1..n] – an  array of integers,
           q – an integer.
OUTPUT:  j s.t. A[j]=q, or -1 if ∀j(1≤j≤n): A[j]≠q

j := 1
while j ≤ n and A[j] != q do j++
if j ≤ n then return j
else return -1
```

- Worst case running time: *n*
- Average case running time: *(n+1)/2* (if *q* is present)

… under which assumption?

Binary Search: Idea

- Search in a sorted array
- Check the element in the middle of the array
- If we have found the search value,
  we are done
- If not, check whether the search value
  has to be in the left or in the right half of the array
- Depending on the check,
  continue with the left or the right half ...

Binary Search, Recursive Version

***INPUT*: A[1..n] – sorted (increasing) array of integers, *q* – integer.**
OUTPUT: an index *j* such that A[*j*] = *q*. *-1*, if $\forall j\ (1 \le j \le n)$: A[*j*] $\neq q$

```
searchRec(A,q)
  searchRecAux(A,q,1,n)

searchRecAux(A,q,l,r)
  m := ⌊(l+r)/2⌋;
  if l > r
     then return -1
  else if A(m) = q
     then return m
  else if q < A(m)
     then return searchRecAux(A,q,l,m-1)
     else return searchRecAux(A,q,m+1,r)
```

Binary Search, Iterative Version

INPUT: **A[1..n] – sorted (increasing) array of integers, *q* – integer.**
OUTPUT: an index *j* such that A[*j*] = *q*. *-1*, if $\forall j$ ($1 \le j \le n$): A[*j*] $\neq$ *q*

```
searchIter(A,q)
  l := 1; r := n;
  m := ⌊(l+r)/2⌋;
  while  l ≤ r and A(m) != q do
    if q < A(m)
       then r:=m-1
       else l:=m+1
    m := ⌊(l+r)/2⌋;
  if l > r
     then return -1
     else return m
```

Analysis of Binary Search

How many times is the loop executed?

- With each execution
  the difference between `l` and `r` is cut in half
  - Initially the difference is *n*
  - *The loop stops when the difference becomes 0 (less than 1)*
- How many times do you have to cut *n* in half to get 0?
- *log n* – better than the brute-force approach of linear search (*n*).

Linear vs Binary Search

- Costs of linear search: n
- Costs of binary search: log(n)
- Should we care?
- Phone book with *n* entries:
  - n = 200,000, log n = log 200,000 = 8 +10
  - n = 2M, log 2M = 1 + 10 + 10
  - n = 20M, log 20M = 5+20

DSA, Part 2: Overview

- Complexity of algorithms
- Asymptotic analysis
- Special case analysis
- Correctness of algorithms

Asymptotic Analysis

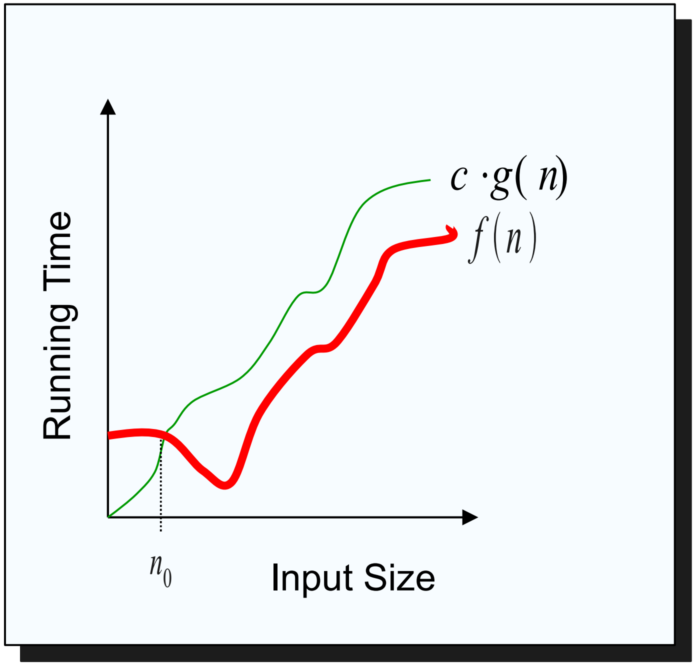
- Goal: simplify the analysis of the running time by getting rid of details, which are affected by specific implementation and hardware
  - "rounding" of numbers: 1,000,001 ≈ 1,000,000
  - "rounding" of functions: $3n^2 \approx n^2$
- Capturing the essence: how the running time of an algorithm increases with the size of the input *in the limit*
  - Asymptotically more efficient algorithms are best for all but small inputs

Asymptotic Notation

The “big-Oh” *O*-Notation

- talks about asymptotic upper bounds
- $f(n) = O(g(n))$ iff there exist $c > 0$ and $n_0 > 0$, s.t. $f(n) \leq c\, g(n)$ for $n \geq n_0$
- $f(n)$ and $g(n)$ are functions over non-negative integers

Used for *worst-case* analysis



Asymptotic Notation, Example

$f(n) = 2n^2 + 3(n+1)$, $g(n) = 3n^2$

Values of $f(n) = 2n^2 + 3(n+1)$:

2+6, 8+9, 18+12, 32+15

Values of $g(n) = 3n^2$:

3, 12, 27, 64

From $n_0 = 4$ onward, we have $f(n) \leq g(n)$

Asymptotic Notation, Examples

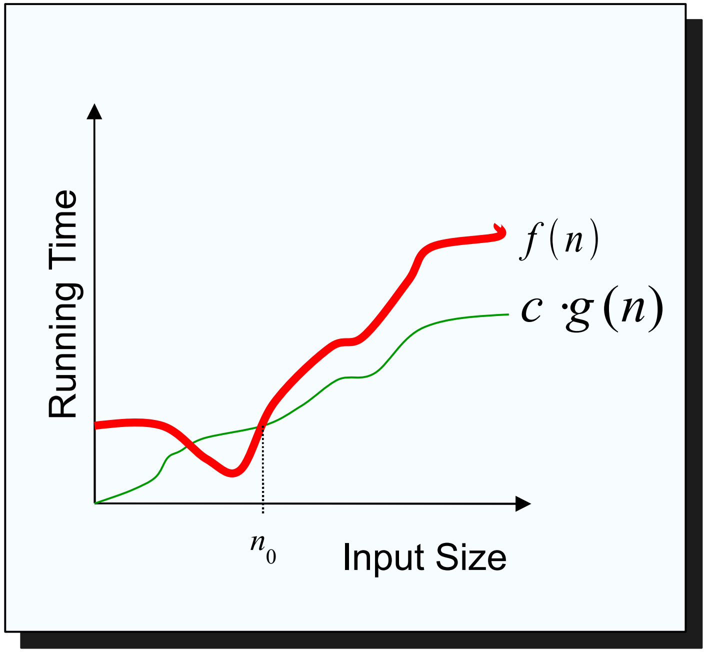
- Simple Rule: We can always drop lower order terms and constant factors, without changing big Oh:
  - $7n + 3$ is $O(n)$
  - $8n^2 \log n + 5n^2 + n$ is $O(n^2 \log n)$
  - $50\, n \log n$ is $O(n \log n)$

- Note:
  - $50\, n \log n$ is $O(n^2)$
  - $50\, n \log n$ is $O(n^{100})$

  but this is less informative than saying
  - $50\, n \log n$ is $O(n\, log\, n)$

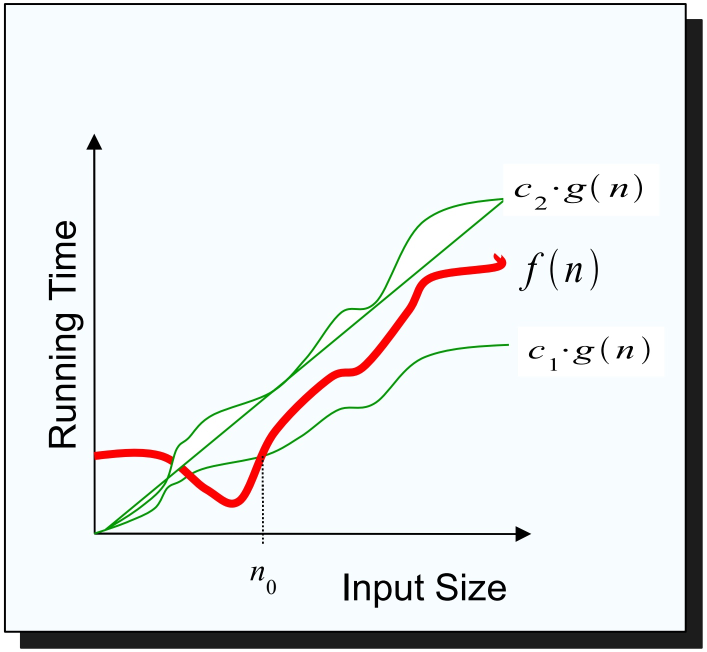
Asymptotic Notation/2

- The "big-Omega" Ω-Notation
  - asymptotic lower bound
  - $f(n) = \Omega(g(n))$ iff
    there exist $c > 0$ and $n_0 > 0$,
    s.t. $c\ g(n) \leq f(n)$, for $n \geq n_0$
- Used to describe lower bounds of algorithmic problems
  - E.g., searching in
    a sorted array
    with linear search is $\Omega(n)$,
    with binary search is $\Omega(\log n)$



Asymptotic Notation/3

- The "big-Theta" Θ-Notation
  - asymptotically tight bound
  - $f(n) = \Theta(g(n))$ if there exists $c_1>0$, $c_2>0$, and $n_0>0$, *s.t.* for $n \geq n_0$ $c_1\, g(n) \leq f(n) \leq c_2\, g(n)$



- $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- *Note: $O(f(n))$* is often used when $\Theta(f(n))$ *is meant*

Asymptotic Notation/4

- Analogy with real numbers
  - $f(n) = O(g(n)) \quad \cong f \leq g$
  - $f(n) = \Omega(g(n)) \quad \cong f \geq g$
  - $f(n) = \Theta(g(n)) \quad \cong f = g$

- Abuse of notation:

  $f(n) = O(g(n))$ actually means

  $f(n) \in O(g(n))$

Exercise: Asymptotic Growth

Order the following functions according to their asymptotic growth.

- $2^n + n^2$
- $3n^3 + n^2 – 2n^3 + 5n – n^3$
- $20 \log_2 2n$
- $20 \log_2 n^2$
- $20 \log_2 4^n$
- $20 \log_2 2^n$
- 3^n

Comparison of Running Times

Determining the maximal problem size

Running Time $T(n)$ in μs	1 second	1 minute	1 hour
$400n$	2,500	150,000	9,000,000
$20n \log n$	4,096	166,666	7,826,087
$2n^2$	707	5,477	42,426
n^4	31	88	244
2^n	19	25	31

DSA, Part 2: Overview

- Complexity of algorithms
- Asymptotic analysis
- Special case analysis
- Correctness of algorithms

Special Case Analysis

- Consider extreme cases and make sure your solution works in all cases.
- The problem: identify special cases.
- This is related to INPUT and OUTPUT specifications.

Special Cases

- empty data structure (array, file, list, …)
- single element data structure
- completely filled data structure
- entering a function
- termination of a function
- zero, empty string
- negative number
- border of domain
- start of loop
- end of loop
- first iteration of loop

Sortedness

The following algorithm checks whether an array is sorted.

```
INPUT: A[1..n] - an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n
  if A[i] ≥ A[i+1] then return FALSE
return TRUE
```

Analyze the algorithm by considering special cases.

Sortedness/2

```
INPUT:  A[1..n] - an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n
  if A[i] ≥ A[i+1] then return FALSE
return TRUE
```

- Start of loop, i=1 ➔ OK
- End of loop, i=n ➔ ERROR (tries to access A[n+1])

Sortedness/3

```
INPUT:  A[1..n] - an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n-1
  if A[i] ≥ A[i+1] then return FALSE
return TRUE
```

- Start of loop, i=1 Ⓟ OK
- End of loop, i=n-1 Ⓟ OK
- A=[1,2,3] Ⓟ First iteration, from i=1 to i=2 Ⓟ OK
- A=[1,2,2] Ⓟ ERROR (if A[i]=A[i+1] for some i)

Sortedness/4

```
INPUT: A[1..n] – an array of integers.
OUTPUT: TRUE if A is sorted; FALSE otherwise

for i := 1 to n-1
  if A[i] > A[i+1] then return FALSE
return TRUE
```

- Start of loop, i=1 ➔ OK
- End of loop, i=n-1 ➔ OK
- A=[1,2,3] ➔ First iteration, from i=1 to i=2 ➔ OK
- A=[1,1,1] ➔ OK
- Empty data structure, n=0 ➔ ? (for loop)
- A=[-1,0,1,-3] ➔ OK

Binary Search, Variant 1

Analyze the following algorithm
by considering special cases.

```
l := 1; r := n
do
   m := ⌊(l+r)/2⌋
   if A[m] = q then return m
   else if A[m] > q then r := m-1
   else l := m+1
while l < r
return -1
```

Binary Search, Variant 1

```
l := 1; r := n
do
  m := ⌊(l+r)/2⌋
  if A[m] = q then return m
  else if A[m] > q then r := m-1
  else l := m+1
while l < r
return -1
```

- Start of loop ➔ OK
- End of loop, l=r ➔ Error! Example: search 3 in [3 5 7]

Binary Search, Variant 1

```
l := 1; r := n
do
   m := ⌊(l+r)/2⌋
   if A[m] = q then return m
   else if A[m] > q then r := m-1
   else l := m+1
while l <= r
return -1
```

- Start of loop ➔ OK
- End of loop, l=r ➔ OK
- First iteration ➔ OK
- A=[1,1,1] ➔ OK
- Empty array, n=0 ➔ Error! Tries to access A[0]
- One-element array, n=1 ➔ OK

Binary Search, Variant 1

```
l := 1; r := n
if r < l then return -1;
do
   m := ⌊(l+r)/2⌋
   if A[m] = q then return m
   else if A[m] > q then r := m-1
   else l := m+1
while l <= r
return -1
```

- Start of loop ➔ OK
- End of loop, l=r ➔ OK
- First iteration ➔ OK
- A=[1,1,1] ➔ OK
- Empty data structure, n=0 ➔ OK
- One-element data structure, n=1 ➔ OK

Binary Search, Variant 2

Analyze the following algorithm
by considering special cases

```
l := 1; r := n
while l < r do
   m := ⌊(l+r)/2⌋
   if A[m] <= q
      then l := m+1 else r := m
if A[l-1] = q
   then return l-1 else return -1
```

Binary Search, Variant 3

Analyze the following algorithm by considering special cases

```
l := 1; r := n
while l <= r do
  m := ⌊(l+r)/2⌋
  if A[m] <= q
    then l := m+1 else r := m
if A[l-1] = q
  then return l-1 else return -1
```

Insertion Sort, Slight Variant

- Analyze the following algorithm by considering special cases
- Hint: beware of lazy evaluations

```
INPUT:   A[1..n] - an array of integers
OUTPUT:  permutation of A s.t.
         A[1] ≤ A[2] ≤ ... ≤ A[n]

for j := 2 to n do
  key := A[j]; i := j-1;
  while A[i] > key and i > 0 do
    A[i+1] := A[i];  i--;
  A[i+1] := key
```

Merge

Analyze the following algorithm
by considering special cases.

```
INPUT:  A[1..n1], B[1..n2] sorted arrays of
        integers, C[1..n1+n2] array

OUTPUT:  permutation C of A.B s.t.
   C[1] ≤ C[2] ≤. . . ≤ C[n1+n2]

i:=1;  j:=1;
for k:=1 to n1 + n2 do
      if A[i] <= B[j]
      then C[k] := A[i]; i++;
      else C[k] := B[j]; j++;
return C;
```

Merge/2

INPUT: A[1..n1], B[1..n2] sorted arrays of integers, C[1..n1+n2] array
OUTPUT: permutation C of A.B s.t.
C[1] ≤ C[2] ≤ . . . ≤ C[n1+n2]

```
i:=1 ;j:=1 ;
for k:= 1 to n1 + n2 do
   if j > n2 or (i <= n1 and A[i]<=B[j])
      then C[k] := A[i]; i++;
      else C[k] := B[j]; j++;
return C;
```

Merge/3

To analyze the algorithm on the previous slide, we distinguish 4 cases

- neither A nor B exhausted implies
  “A[i] ≤ B[j]” decides
- B exhausted, A not, implies
  $j > n2$, implies A wins
- B not exhausted, A exhausted, implies
  $j \leq n2$ && $i > n$, implies B wins
- A, B both exhausted, implies
  $k > n1+n2$, implies algorithm stops

DSA, Part 2: Overview

- Complexity of algorithms
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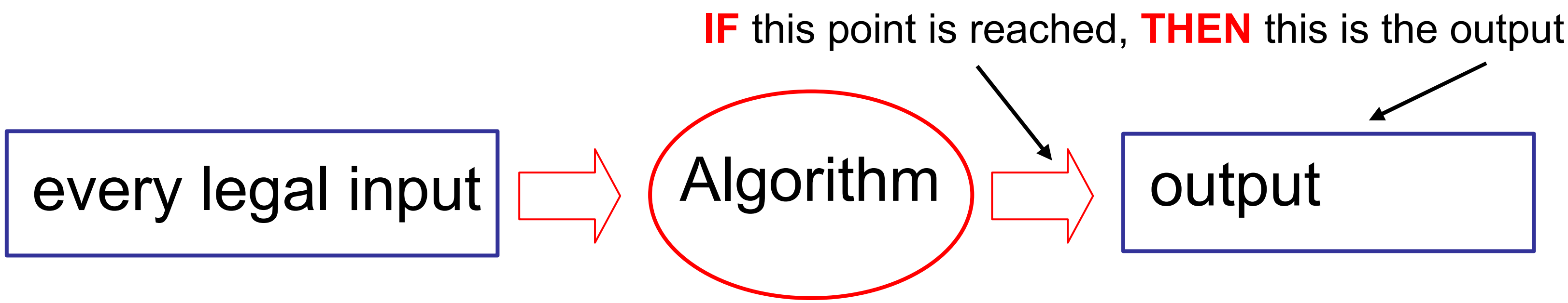
- Special case analysis
- Correctness of algorithms

Correctness of Algorithms

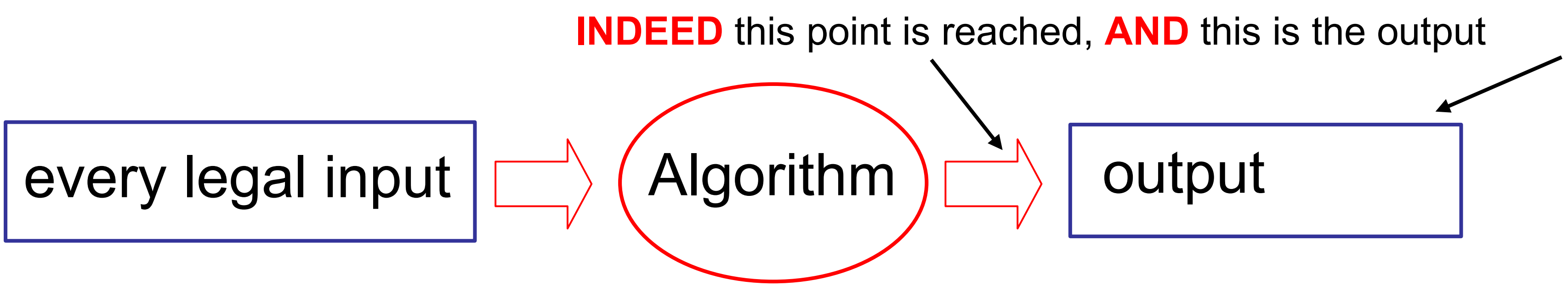
- An algorithm is *correct* if for every legal input, it terminates and produces the desired output.
- Automatic proof of correctness is not possible (this is one of the so-called "undecidable problems")
- There are practical techniques and rigorous formalisms that help one to reason about the correctness of (parts of) algorithms.

Partial and Total Correctness

- Partial correctness



- Total correctness



Assertions

- To prove partial correctness we associate a number of assertions (statements about the state of the execution) with specific checkpoints in the algorithm.
  - E.g., "*A*[1], …, *A*[j] form an increasing sequence"
- **Preconditions** – assertions that must be valid *before* the execution of an algorithm or a subroutine (INPUT)
- **Postconditions** – assertions that must be valid *after* the execution of an algorithm or a subroutine (OUTPUT)

Pre- and Postconditions of Linear Search

```
INPUT:  A[1..n] - a array of integers,
              q - an integer.
OUTPUT:  j s.t. A[j]=q. -1 if ∀i(1≤i≤n): A[i]≠q

j := 1
while j ≤ n and A[j] != q do j++
if j ≤ n then return j
  else return -1
```

How can we be sure that

- whenever the precondition holds,
- also the postcondition holds?

Loop Invariant in Linear Search

```
j := 1
while j ≤ n and A[j] != q do j++
if j ≤ n then return j
   else return -1
```

Whenever the beginning of the loop is reached, then

$A[i] \neq q$ for all i where $1 \leq i < j$

When the loop stops, there are two cases

- $j = n+1$, which implies $A[i] \neq q$ for all i, $1 \leq i < n+1$
- $A[j] = q$

Loop Invariant in Linear Search

```
j := 1
while j ≤ n and A[j] != q do j++
if j ≤ n then return j
else return -1
```

Note: The condition

$A[i] \mathrel{!}= q$ for all i where $1 \leq i < j$

- holds when the loop is entered for the first time
- continues to hold until we reach the loop for the last time

Loop Invariants

- **Invariants:** assertions that are valid every time the beginning of the loop is reached (many times during the execution of an algorithm)
- We must show three things about loop invariants:
  - **Initialization:** it is true prior to the first iteration.
  - **Maintenance:** *if* it is true before an iteration, *then* it is true after the iteration.
  - **Termination:** when a loop terminates, the invariant gives a useful property to show the correctness of the algorithm

Example: Binary Search/1

```
l:= 1; r:= n;
m:= ⌊(l+r)/2⌋;
while l <= r and A(m) != q do
   if q < A(m)
      then r:=m-1
      else l:=m+1
    m:=⌊(l+r)/2⌋;
if l > r
    then return -1
    else return m
```

- We want to show that q is not in A if -1 is returned.
- **Invariant:**
  $\forall i \in [1..l-1]: A[i]<q$ (ia)
  $\forall i \in [r+1..n]: A[i]>q$ (ib)
- **Initialization**: $l = 1$, $r = n$
  the invariant holds because there are no elements to the left of *l* or to the right of *r*.

  $l = 1$ yields $\forall\, i \in [1..0]: A[i]<q$
  this holds because [1..0] is empty

  $r = n$ yields $\forall\, i \in [n+1..n]: A[i]>q$
  this holds because [n+1..n] is empty

Example: Binary Search/2

```
l:= 1; r:= n;
m:= ⌊(l+r)/2⌋;
while l <= r and A(m) != q do
  if q < A(m)
     then r:=m-1
     else l:=m+1
   m := ⌊(l+r)/2⌋;
if l > r
   then return -1
   else return m
```

- **Invariant:**
  $\forall i \in [1..l-1]$: $A[i]<q$ (ia)
  $\forall i \in [r+1..n]$: $A[i]>q$ (ib)

- **Maintenance**: $1 \leq l$, $r \leq n$, $m = \lfloor (l+r)/2 \rfloor$
  We consider two cases:
  - A[m] != q & q < A[m]: implies $r = m-1$
    A sorted implies $\forall k \in [r+1..n]$: $A[k] > q$ (ib)
  - A[m] != q & A[m] < q: implies $l = m+1$
    A sorted implies $\forall k \in [1..l-1]$: $A[k] < q$ (ia)

Example: Binary Search/3

```
l:= 1; r:= n;
m:= ⌊(l+r)/2⌋;
while l <= r and A(m) != q do
  if q < A(m)
      then r:=m-1
       else l:=m+1
    m := ⌊(l+r)/2⌋;
if l > r
    then return -1
    else return m
```

- **Invariant:**
  $\forall i \in [1..l-1]: A[i] < q$ (ia)
  $\forall i \in [r+1..n]: A[i] > q$ (ib)

- **Termination**: $1 \le l,\ r \le n,\ l \le r$

  Two cases:

  $l := m+1$ implies $l_{new} = \lfloor (l+r)/2 \rfloor + 1 > l_{old}$

  $r := m-1$ implies $r_{new} = \lfloor (l+r)/2 \rfloor - 1 < r_{old}$

- The range gets smaller during each iteration and the loop will terminate when $l \le r$ no longer holds

Example: Insertion Sort/1

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Loop invariants:

External “for” loop

Let A^{orig} denote the array at the beginning of the for loop:

A[1..j-1] is sorted

$A[1..j-1] \in A^{orig}[1..j-1]$

Internal “while” loop

Let A^{orig} denote the array at beginning of the while loop:

- $A[1..i] = A^{orig}[1..i]$
- $A[i+2..j] = A^{orig}[i+1..j-1]$
- A[k] > key for all k in {i+2,…,j}

Example: Insertion Sort/2

External for loop:

(i) A[1...j-1] is sorted

(ii) A[1...j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1..i] = A^{orig}[1..i]
- A[i+2..j] = A^{orig}[i+1..j-1]
- A[k] > key for all k in {i+2,...,j}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Initialization:

External loop: (i), (ii) j = 2: A[1..1] $\in$ A^{orig}[1..1] and is trivially sorted

Internal loop: i = j-1:

- A[1...j-1] = A^{orig}[1..j-1] , since nothing has happend
- A[j+1..j] = A^{orig}[j..j-1] , since both sides are empty
- A[k] > key holds trivially for all k in the empty set

Example: Insertion Sort/3

External for loop:

(i) A[1..j-1] is sorted

(ii) A[1..j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1..i] = A^{orig}[1..i]
- A[i+2..j] = A^{orig}[i+1..j-1]
- A[k] > key for all k in {i+2,...,j}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Maintenance internal while loop

Before the decrement "`i--`", the following facts hold:

- A[1..i-1] = A^{orig}[1..i-1] (because nothing in A[1..i-1] has been changed)
- A[i+1..j] = A^{orig}[i..j-1] (because A[i] has been copied to A[i+1] and A[i+2..j] = A^{orig}[i+1..j-1]
- A[k] > key for all k in {i+1,...,j} (because A[i] has been copied to A[i+1])

After the decrement "`i--`", the invariant holds because i-1 is replaced by i.

Example: Insertion Sort/4

External for loop:

(i) A[1..j-1] is sorted

(ii) A[1..j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1..i] = A^{orig}[1..i]
- A[i+2..j] = A^{orig}[i+1..j-1]
- key < A[k] for all k in {i+2,...,j}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Termination internal while loop

The while loop terminates, since i is decremented in each round.

Termination can be due to two reasons:

i=0: A[2..j] = A^{orig}[1..j-1] and key < A[k] for all k in {2,...,j} (because of the invariant)
implies key, A[2..j] is a sorted version of A^{orig}[1..j]

A[i] $\leq$ key: A[1..i] = A^{orig}[1..i], A[i+2..j] = A^{orig}[i+1..j-1], key = A^{orig}[j]
implies A[1..i], key, A[i+2..j] is a sorted version of A^{orig}[1..j]

Example: Insertion Sort/5

External for loop:

(i) A[1..j-1] is sorted

(ii) A[1..j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1..i] = A^{orig}[1..i]
- A[i+2..j] = A^{orig}[i+1..j-1]
- key < A[k] for all k in {i+2,...,j}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Maintenance of external for loop

When the internal while loop terminates, we have (see previous slide):

A[1..i], key, A[i+2..j] is a sorted version of A^{orig}[1..j]

After

- assigning key to A[i+1] and
- Incrementing j,

the invariant of the external loop holds again.

Example: Insertion Sort/6

External for loop:

(i) A[1..j-1] is sorted

(ii) A[1..j-1] $\in$ A^{orig}[1..j-1]

Internal while loop:

- A[1..i] = A^{orig}[1..i]
- A[i+2..j] = A^{orig}[i+1..j-1]
- key < A[k]   for all k in {i+2,...,j}

```
for j := 2 to n do
  key := A[j]
  i := j-1
  while i>0 and A[i]>key do
    A[i+1] := A[i]
    i--
  A[i+1] := key
```

Termination of external for loop

The for loop terminates because j is incremented in each round.

Upon termination, j = n+1 holds.

In this situation, the loop invariant of the for loop says:

A[1..n] is sorted and contains the same values as A^{orig}[1..n]

That is, A has been sorted.

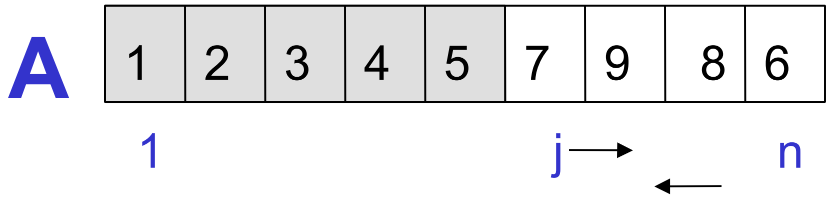
Example: Bubble Sort

```
INPUT: A[1..n] – an array of integers
OUTPUT: permutation of A s.t. A[1] ≤ A[2] ≤ … ≤ A[n]

for j := 1 to n-1 do
  for i := n downto j+1 do
    if A[i-1] > A[i] then
      swap(A,i-1,i)
```

- What is a good loop invariant for the outer loop?
  (i.e., a property that always holds at the end of line 1)
- … and what is a good loop invariant for the inner loop?
  (i.e., a property that always holds at the end of line 2)

Example: Bubble Sort



Strategy

- Start from the back and compare pairs of adjacent elements.
- Swap the elements if the larger comes before the smaller.
- In each step the smallest element of the unsorted part is moved to the beginning of the unsorted part and the sorted part grows by one.

44	55	12	42	94	18	06	67
06	44	55	12	42	94	18	67
06	12	44	55	18	42	94	67
06	12	18	44	55	42	67	94
06	12	18	42	44	55	67	94
06	12	18	42	44	55	67	94
06	12	18	42	44	55	67	94
06	12	18	42	44	55	67	94

Loop Invariants for Bubble Sort

- Outer loop: “A[1..j-1] is sorted and contains
  the j-1 smallest values of the array”

  Note: loop finishes with j = n

  In the end:

  A[1..n-1] is sorted and minimum,

  hence, A[1..n] is sorted
- Inner loop: “A[i] is the minimum in A[i..n]”

  Note: loop finishes with i = j

  In the end:

  A[j] is the minimum in A[j..n],
  which implies the outer loop invariant

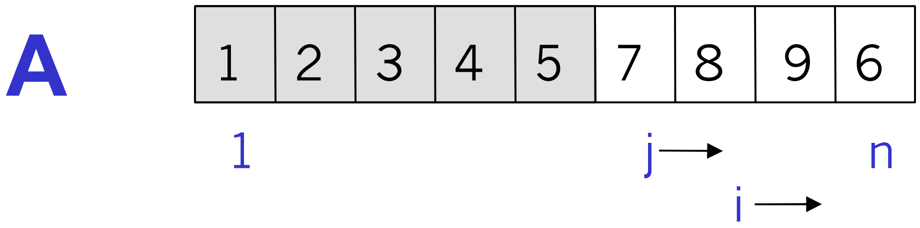
Example: Selection Sort

```
INPUT: A[1..n] – an array of integers
OUTPUT: a permutation of A such that A[1]≤A[2]≤…≤A[n]

for j := 1 to n-1 do
  min := A[j]; minpos := j
  for i := j+1 to n do
    if A[i] < min then min := A[i]; minpos := i;
  A[minpos] := A[j]; A[j] := min
```

- What is a good loop invariant for the outer loop?
- … and what is a good loop invariant for the inner loop?

Example: Selection Sort



Strategy

- Start empty handed.
- Enlarge the sorted part by swapping the first element of the unsorted part with the smallest element of the unsorted part.
- Continue until the unsorted part consists of one element only.

44	55	12	42	94	18	06	67
06	55	12	42	94	18	44	67
06	12	55	42	94	18	44	67
06	12	18	42	94	55	44	67
06	12	18	42	94	55	44	67
06	12	18	42	44	55	94	67
06	12	18	42	44	55	94	67
06	12	18	42	44	55	67	94

Loop Invariants for Selection Sort

- Outer loop: “A[1..j-1] sorted and contains the j-1 smallest values of the array”

  Note: loop finishes with j = n

  In the end: A[1..n-1] is sorted and minimum, hence, A[1..n] is sorted

- Inner loop: “min holds the minimum of A[j..i-1] and minpos holds the position of the minimum”

  Note: loop finishes with i = n+1

  In the end: min holds the minimum of A[j..n] then, swap(minpos,j) puts min into j, which implies the outer loop invariant

Exercise

- Apply the same approach that we used for insertion sort to prove the correctness of bubble sort and selection sort.

Math Refresher

- Arithmetic progression

$$\sum_{i=0}^{n} i = 0 + 1 + \ldots + n = n(n+1)/2$$

- Geometric progression (for a number a ≠ 1)

$$\sum_{i=0}^{n} a^i = 1 + a^2 + \ldots + a^n = (1 - a^{n+1})/(1 - a)$$

Induction Principle

We want to show that property P is true for all integers $n \geq n_0$.

Basis: prove that P is true for n_0.

Inductive step: prove that if P is true for all k such that $n_0 \leq k \leq n - 1$ then P is also true for n.

Exercise: Prove that every Fibonacci number of the form fib(3n) is even

Summary

- Algorithmic complexity
- Asymptotic analysis
  - Big O and Theta notation
  - Growth of functions and asymptotic notation
- Correctness of algorithms
  - Pre/Post conditions
  - Invariants
- Special case analysis